

GROUPOID COHOMOLOGY AND EXTENSIONS

BY J. L. TU

ABSTRACT. We show that Haefliger's cohomology for étale groupoids, Moore's cohomology for locally compact groups and the Brauer group of a locally compact groupoid are all particular cases of sheaf (or Čech) cohomology for topological simplicial spaces.

CONTENTS

1. Introduction	1
2. Simplicial spaces and groupoids	3
2.1. Definition of simplicial spaces	3
2.2. Groupoids	4
2.3. Morita equivalence and generalized morphisms	5
3. Sheaves on simplicial spaces	6
3.1. Basic definitions	6
3.2. G -sheaves and sheaves over simplicial spaces	8
4. Čech cohomology	9
4.1. Covers of simplicial spaces	9
4.2. Čech cohomology	11
4.3. Compatibility with usual Čech cohomology for spaces	16
4.4. Long exact sequences in Čech cohomology	17
5. Low dimensional Čech cohomology	18
5.1. The group \check{H}^0	18
5.2. The group \check{H}^1	18
5.3. The group \check{H}^2 , extensions and the Brauer group	19
6. Comparison with Moore's cohomology	25
7. Comparison with sheaf cohomology and Haefliger's cohomology	27
8. Invariance by Morita equivalence	30
References	31

1. INTRODUCTION

Let G be a locally compact Hausdorff groupoid with Haar system. In [10], the authors studied the group of Morita equivalence classes of actions of G on continuous fields of C^* -algebras over the unit space G_0 such that

- Each fiber is isomorphic to the algebra of compact operators on some Hilbert space (depending on the fiber);

- The bundle satisfies Fell's condition, i.e. each point of G_0 has a neighborhood U such that there exists a section $f(x)$ with $f(x)$ a rank-one projection for all $x \in U$.

They called this group the Brauer group $Br(G)$ of G , and showed that it is naturally isomorphic to the group $\text{Ext}(G, \mathbb{T})$ of Morita equivalence classes of central extensions

$$\mathbb{T} \times G'_0 \rightarrow E \rightarrow G',$$

where G' is some Morita equivalent groupoid. In the case of discrete groups, it is well-known that central extensions of G by \mathbb{T} are classified by $H^2(G, \mathbb{T})$. Actually, given any locally compact group G and any Polish (i.e. metrizable separable complete) G -module A , Moore's cohomology groups $H^2(G, A)$ classify extensions $A \hookrightarrow E \twoheadrightarrow G$ such that the action of G on A by conjugation is exactly the action of G on the G -module A [15, 16]. One of the possible definitions of Moore's cohomology is the following: consider $C^n(G, A)$ the space of all measurable maps $c: G^n \rightarrow A$ with the differential

$$(1.1) \quad (dc)(g_1, \dots, g_n) = g_1 c(g_2, \dots, g_n) + \sum_{k=1}^n (-1)^k c(g_1, \dots, g_k g_{k+1}, \dots, g_n) + (-1)^{n+1} c(g_1, \dots, g_n),$$

then $H^n(G, A)$ is the n -th cohomology group of the complex $C^*(G, A)$.

On the other hand, Haefliger ([6], see also [9]) defined sheaf cohomology groups $H^*(G, \mathcal{A})$ given any étale groupoid G and any abelian G -sheaf \mathcal{A} (i.e. an abelian sheaf on G_0 endowed with a continuous action of G). It was thus natural to expect that a single cohomology theory for groupoids should unify all these: this is the question asked by A. Kumjian in Boulder (1999).

Our approach is to consider the simplicial (topological) space $G_\bullet = (G_n)_{n \in \mathbb{N}}$ and to use sheaf cohomology for simplicial spaces [4] and Čech cohomology (see Section 4). We show:

Theorem 1.1. *Let M_\bullet be a simplicial space and \mathcal{A}^\bullet be an abelian sheaf on M_\bullet . Denote by $H^n(M_\bullet; \mathcal{A}^\bullet)$ and $\check{H}(M_\bullet; \mathcal{A}^\bullet)$ the sheaf and Čech cohomology groups respectively. Then*

- $H^n(M_\bullet; \mathcal{A}^\bullet) \cong \check{H}(M_\bullet; \mathcal{A}^\bullet)$ if M_n is paracompact for all n ;
- $H^n(G_\bullet; \mathcal{A}^\bullet)$ coincides with Haefliger's cohomology $H^n(G; \mathcal{A}^0)$ if G is an étale groupoid and \mathcal{A}^\bullet is the sheaf on G_\bullet corresponding to an abelian G -sheaf \mathcal{A}^0 ;
- $H^n(G_\bullet; \mathcal{A}^\bullet) \cong \check{H}(G_\bullet; \mathcal{A}^\bullet) \cong H_{\text{Moore}}^n(G, A)$ if G is a locally compact group, A is a Polish G -module and \mathcal{A}^\bullet is the sheaf on G_\bullet associated to A ;
- $H^2(G_\bullet; \mathbb{T}) \cong \check{H}(G_\bullet; \mathbb{T})$ is the Brauer group of G if G is a locally compact Hausdorff groupoid with Haar system;
- $H^n(G_\bullet, \mathcal{A}^\bullet)$ and $\check{H}^*(G_\bullet, \mathcal{A}^\bullet)$ are invariant under Morita equivalence of topological groupoids.

(Actually we prove more than statement (d); see Proposition 5.6.)

It is possible that some parts of this paper are well-known among specialists, but apparently written nowhere in the literature, in particular the definition of Čech cohomology and its relation to sheaf cohomology for simplicial spaces. Besides, a more conceptual approach would have consisted in using Grothendieck's cohomology of sites [1], as Moerdijk did in [13]. Moreover, it is possible that the present work has non-trivial intersection with Moerdijk's in [14]. However, we hope that the present approach, being rather direct and elementary, will still be of interest to the reader.

Acknowledgments: the author would like to thank Ping Xu for useful discussions and Kai Behrend for providing some bibliographic references.

2. SIMPLICIAL SPACES AND GROUPOIDS

2.1. Definition of simplicial spaces. Let us recall some basic facts about simplicial spaces. Let Δ (resp. Δ') be the simplicial (resp. semi-simplicial) category, whose objects are the nonnegative integers, and whose morphisms are the nondecreasing (resp. increasing) maps $[m] \rightarrow [n]$ (where $[n]$ denotes the interval $\{0, \dots, n\}$). We denote by $\Delta^{(N)}$ the N -truncated simplicial category, i.e. the full sub-category of Δ whose objects are the integers $\leq N$.

A simplicial (resp. semi-simplicial, N -simplicial) topological space is a contravariant functor from the category Δ (resp. Δ' , resp. $\Delta^{(N)}$) to the category of topological spaces. In the same way, one can define the notion of simplicial (resp. semi-simplicial, N -simplicial) manifold. In this paper we shall work with simplicial topological spaces and will use the terminology "simplicial space", but some of the results can easily be transposed to simplicial manifolds.

In practice, a (semi-)simplicial space is a sequence $M_\bullet = (M_n)_{n \in \mathbb{N}}$ of topological spaces, given with continuous maps $\tilde{f}: M_n \rightarrow M_k$ for every morphism $f: [k] \rightarrow [n]$, satisfying the relation $\tilde{f} \circ g = \tilde{g} \circ \tilde{f}$ for all composable morphisms f and g .

Let $\varepsilon_i^n: [n-1] \rightarrow [n]$ be the unique increasing map that avoids i , and $\eta_i^n: [n+1] \rightarrow [n]$ be the unique non-decreasing surjective map such that i is reached twice ($0 \leq i \leq n$). We will usually omit the superscripts for convenience of notation.

If $M_\bullet = (M_n)_{n \in \mathbb{N}}$ is a simplicial space, then the face maps $\tilde{\varepsilon}_i^n: M_n \rightarrow M_{n-1}$, $i=0, \dots, n$ and the degeneracy maps $\tilde{\eta}_i^n: M_n \rightarrow M_{n+1}$, $i=0, \dots, n$, satisfy the following simplicial identities: $\tilde{\varepsilon}_i^{n-1} \tilde{\varepsilon}_j^n = \tilde{\varepsilon}_{j-1}^{n-1} \tilde{\varepsilon}_i^n$ if $i < j$, $\tilde{\eta}_i^{n+1} \tilde{\eta}_j^n = \tilde{\eta}_{j+1}^{n+1} \tilde{\eta}_i^n$ if $i \leq j$, $\tilde{\varepsilon}_i^{n+1} \tilde{\eta}_j^n = \tilde{\eta}_{j-1}^{n-1} \tilde{\varepsilon}_i^n$ if $i < j$, $\tilde{\varepsilon}_i^{n+1} \tilde{\eta}_j^n = \tilde{\eta}_j^{n-1} \tilde{\varepsilon}_{i-1}^n$ if $i > j+1$ and $\tilde{\varepsilon}_j^{n+1} \tilde{\eta}_j^n = \tilde{\varepsilon}_{j+1}^{n+1} \tilde{\eta}_j^n = \text{Id}_{M_n}$.

Conversely, if we are given a sequence M_\bullet of topological spaces and maps satisfying such identities, then there is a unique simplicial structure on M_\bullet such that $\tilde{\varepsilon}_i^n$ are the face maps and $\tilde{\eta}_i^n$ are the degeneracy maps.

2.2. Groupoids. In order to fix notations, we first recall some basic facts about groupoids. For more details, see e.g. [18].

A topological groupoid is given by two topological spaces G_0 and G , two maps r and s from G to G_0 , called the range and source maps, a unit map $\eta: G_0 \rightarrow G$, a partially defined multiplication $G_2 = \{(g, h) \in G^2 \mid s(g) = t(h)\} \rightarrow G$ denoted by $(g, h) \mapsto gh$, and an inversion map $G \rightarrow G$ denoted by $g \mapsto g^{-1}$ such that the following identities hold (for $g, h, k \in G$ and $x \in G_0$):

- $r(gh) = r(g)$, $s(gh) = s(h)$;
- $(gh)k = g(hk)$ whenever $s(g) = r(h)$ and $s(h) = r(k)$;
- $s(\eta(x)) = r(\eta(x)) = x$;
- $g\eta(s(g)) = \eta(r(g))g = g$;
- $r(g^{-1}) = s(g)$, $s(g^{-1}) = r(g)$, $gg^{-1} = \eta(r(g))$, $g^{-1}g = \eta(s(g))$.

We will usually identify the unit space G_0 to a subspace of G by means of the unit map η .

Standard examples are:

- groups, with $G_0 = \text{pt}$;
- spaces M , with $G = G_0 = M$, $r = s = \text{Id}$;
- the homotopy groupoid of a space M , where $G_0 = M$, G is the set of homotopy classes of paths in M , $s(g)$ is the starting point of the path and $r(g)$ is the endpoint.

Here are a few notations that we will use: $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, $G_x^y = G_x \cap G^y$.

A left *action* of a topological groupoid G on a space Z is given by a (continuous) map $p: Z \rightarrow G_0$ and a map $G \times_{s,p} Z \rightarrow Z$, denoted by $(g, z) \mapsto gz$, such that

- $p(gz) = r(g)$;
- $(gh)z = g(hz)$ whenever $(g, h) \in G_2$ and $s(h) = p(z)$;
- $ez = z$ if $e \in G_0 \subset G$.

We will say that Z is a (left) G -space. Given a G -space Z , we form the crossed-product groupoid $G \ltimes Z := G \times_{s,p} Z$ with unit space Z , source and range maps $s(g, z) = z$, $r(g, z) = gz$, product $(g, z)(h, z') = (gh, z')$ if $z = hz'$, and inverse $(g, z)^{-1} = (g^{-1}, gz)$.

Any topological groupoid G canonically gives rise to a simplicial space as follows [19]: Let

$$G_n = \{(g_1, \dots, g_n) \mid s(g_i) = t(g_{i+1}) \forall i\}$$

be the set composable n -tuples.

Define the face maps $\tilde{\varepsilon}_i^n: G_n \rightarrow G_{n-1}$ by, for $n > 1$

$$\begin{aligned} \tilde{\varepsilon}_0^n(g_1, g_2, \dots, g_n) &= (g_2, \dots, g_n) \\ \tilde{\varepsilon}_n^n(g_1, g_2, \dots, g_n) &= (g_1, \dots, g_{n-1}) \\ \tilde{\varepsilon}_i^n(g_1, \dots, g_n) &= (g_1, \dots, g_i g_{i+1}, \dots, g_n), \quad 1 \leq i \leq n-1, \end{aligned}$$

and for $n=1$ by, $\tilde{\varepsilon}_0^1(g)=s(g)$, $\tilde{\varepsilon}_1^1(g)=r(g)$. Also define the degeneracy maps: $\tilde{\eta}_0^0 : G_0 \rightarrow G_1$ is the unit map of the groupoid, and $\tilde{\eta}_i^n : G_n \rightarrow G_{n+1}$ by:

$$\begin{aligned}\tilde{\eta}_0^n(g_1, \dots, g_n) &= (r(g_1), g_1, \dots, g_n) \\ \tilde{\eta}_i^n(g_1, \dots, g_n) &= (g_1, \dots, g_i, s(g_i), g_{i+1}, \dots, g_n), \quad 1 \leq i \leq n.\end{aligned}$$

Another way to view the simplicial structure of G_\bullet is the following: we note that G_n can be identified with the quotient of $(EG)_n := \{(\gamma_0, \dots, \gamma_n) \in G^{n+1} \mid r(\gamma_0) = \dots = r(\gamma_n)\}$ by the left action of G , the correspondence being

$$\begin{aligned}(g_1, \dots, g_n) &= (\gamma_0^{-1} \gamma_1, \dots, \gamma_{n-1}^{-1} \gamma_n) \\ [\gamma_0, \dots, \gamma_n] &= [r(g_1), g_1, g_1 g_2, \dots, g_1 \dots g_n].\end{aligned}$$

Then, for any morphism $f: [k] \rightarrow [n]$, $\tilde{f}: G_n \rightarrow G_k$ is defined by

$$\tilde{f}[\gamma_0, \dots, \gamma_n] = [\gamma_{f(0)}, \dots, \gamma_{f(n)}].$$

For instance, in the first picture, if f is injective then

$$(2.1) \quad \tilde{f}(g_1, \dots, g_n) = (g_{f(0)+1} \dots g_{f(1)}, g_{f(1)+1} \dots g_{f(2)}, \dots, g_{f(k-1)+1} \dots g_{f(k)}).$$

2.3. Morita equivalence and generalized morphisms. We recall (see for instance [7, 8, 12, 17, 11, 20]) that a generalized morphism between two (topological, or locally compact, or Lie) groupoids G' and G is given by a topological space (or a locally compact space, or a manifold) Z , two maps $G'_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} G_0$ such that Z admits a left action of G' with respect to ρ , a right action of G with respect to σ , with the property that the two actions commute and $\rho: Z \rightarrow G'_0$ is a locally trivial G' -principal bundle.

Topological (or locally compact...) groupoids and generalized morphisms form a category whose isomorphisms are Morita equivalences. Every groupoid morphism naturally defines a generalized morphism.

If $\mathcal{U} = (U_i)_{i \in I}$ is an open cover of G_0 , define the cover groupoid

$$(2.2) \quad G[\mathcal{U}] = \{(i, g, j) \in I \times G \times I \mid r(g) \in U_i, s(g) \in U_j\}$$

with unit space $\{(i, x) \in I \times G_0 \mid x \in U_i\}$, source and range maps $s(i, g, j) = (j, s(g))$, $r(i, g, j) = (i, r(g))$ and product $(i, g, j)(j, h, k) = (i, gh, k)$.

Then the canonical morphism $G[\mathcal{U}] \rightarrow G$ is a Morita equivalence. Moreover, every generalized morphism $G' \rightarrow G$ admits a decomposition $G' \xleftarrow{\sim} G'[\mathcal{U}'] \xrightarrow{f} G$ for some open cover \mathcal{U}' of G'_0 and some *groupoid morphism* f .

Remark 2.1. The simplicial space $G[\mathcal{U}]_\bullet$ is isomorphic to the sub-simplicial space of $(I^{n+1} \times G_n)_{n \in \mathbb{N}}$ such that $G[\mathcal{U}]_n$ consists of $(2n+1)$ -tuples $(i_0, \dots, i_n, g_1, \dots, g_n)$ satisfying the condition

$$(2.3) \quad r(g_1) \in U_{i_0}, s(g_1) \in U_{i_1}, \dots, s(g_n) \in U_{i_n}.$$

Finally, we will need the following

Proposition 2.2. *For any functor F from the category of topological groupoids to any category, the following are equivalent:*

- (i) F is invariant under Morita-equivalence;
- (ii) F factors through the category whose objects are groupoids and whose morphisms are generalized morphisms;
- (iii) For any groupoid G and any open cover \mathcal{U} of G_0 , the canonical map $G[\mathcal{U}] \rightarrow G$ induces an isomorphism $F(G[\mathcal{U}]) \xrightarrow{\sim} F(G)$.

Proof. See for instance [20, Proposition 2.5]. \square

3. SHEAVES ON SIMPLICIAL SPACES

3.1. Basic definitions. Recall [4] that if $u: X \rightarrow Y$ is continuous, \mathcal{A} is a sheaf on X and \mathcal{B} is a sheaf on Y , then a u -morphism from \mathcal{B} to \mathcal{A} is by definition an element of $\text{Hom}(\mathcal{B}, u_*\mathcal{A}) \cong \text{Hom}(u^*\mathcal{B}, \mathcal{A})$. A sheaf on a simplicial (resp. semi-simplicial) space M_\bullet is a sequence $\mathcal{A}^\bullet = (\mathcal{A}^n)_{n \in \mathbb{N}}$ such that \mathcal{A}^n is a sheaf on M_n , and such that for each morphism $f: [k] \rightarrow [n]$ in the category Δ (resp. Δ') we are given \tilde{f} -morphisms

$$(3.1) \quad \tilde{f}^*: \mathcal{A}^k \rightarrow \mathcal{A}^n$$

such that $\tilde{f}^*\tilde{g}^* = \widetilde{f \circ g}^*$ if $g: [\ell] \rightarrow [k]$

In practice, given open sets $U \subset M_n$ and $V \subset M_k$ such that $\tilde{f}(U) \subset V$ we have a *restriction map* $\tilde{f}^*: \mathcal{A}^k(V) \rightarrow \mathcal{A}^n(U)$ such that $\tilde{f}^*\tilde{g}^* = \widetilde{f \circ g}^*: \mathcal{A}^\ell(W) \rightarrow \mathcal{A}^n(U)$ whenever $\tilde{g}(V) \subset W$.

A fundamental example is given by G -sheaves. In the definition below, recall that a map $f: X \rightarrow Y$ is said to be étale if it is a local homeomorphism, i.e. every point $x \in X$ has an open neighborhood U such that $f(U)$ is open and f induces a homeomorphism from U onto $f(U)$. We will also say that X is an étale space over Y . A groupoid is étale if the range (equivalently the source) map is étale. A morphism $\pi_\bullet: E_\bullet \rightarrow M_\bullet$ is étale if each $\pi_n: E_n \rightarrow M_n$ is étale. Finally, recall that a sheaf over a space X can be considered as a (not necessarily Hausdorff) étale space over X .

Definition 3.1. [6] Let G be a topological groupoid. Then a G -sheaf is an étale space E_0 over G_0 , endowed with a continuous action of G .

Of course, an *abelian* G -sheaf is a G -sheaf E_0 such that E_0 is an abelian sheaf on G_0 and such that for each $g \in G$, the action $\alpha_g: (E_0)_{s(g)} \rightarrow (E_0)_{r(g)}$ is a group morphism.

Example 3.2. If G is a group then a G -sheaf is just a space endowed with an action of G .

To show that any G -sheaf defines a sheaf over G_\bullet , we need some preliminaries:

Definition 3.3. Let $\pi_\bullet: E_\bullet \rightarrow M_\bullet$ a morphism of simplicial spaces. We say that π_\bullet is *reduced* if for all k, n and all $f \in \text{Hom}_\Delta(k, n)$, the map \tilde{f} induces an isomorphism $E_n \cong M_n \times_{\tilde{f}, \pi_k} E_k$. In this case, we will say that E_\bullet is a reduced simplicial space over M_\bullet .

Definition 3.4. Let \mathcal{A}^\bullet be a sheaf over the simplicial space M_\bullet . We will say that \mathcal{A}^\bullet is *reduced* if for all k, n and all $f \in \text{Hom}_\Delta(k, n)$, the morphism $\tilde{f}^* \in \text{Hom}(\tilde{f}^* \mathcal{A}^k, \mathcal{A}^n)$ is an isomorphism.

Lemma 3.5. *There is a one-to-one correspondence between reduced sheaves over M_\bullet and reduced étale simplicial spaces over M_\bullet .*

Proof. The proof is easy. Let us just explain the construction of the sheaf \mathcal{A}^\bullet out of the reduced simplicial space E_\bullet over M_\bullet .

Let $\mathcal{A}^n(U)$ be the space of continuous sections over U of the projection map $\pi_n: E_n \rightarrow M_n$. If $f: [k] \rightarrow [n]$ is a morphism in Δ and $\tilde{f}(U) \subset V$, then for any section $\sigma \in \mathcal{A}^k(V)$ we define $\tilde{f}^* \sigma \in \mathcal{A}^n(U)$ by $(\tilde{f}^* \sigma)(x) = (x, \sigma(\tilde{f}(x))) \in M_n \times_{\tilde{f}, \pi_k} E_k \cong E_n$. \square

Lemma 3.6. *Any reduced simplicial space over M_\bullet , étale or not, determines a sheaf over M_\bullet .*

Proof. The proof is the same. Note that it is not clear whether all sheaves can be constructed this way. \square

Corollary 3.7. *Let G be a topological groupoid, then any G -space determines a sheaf on G_\bullet . If the G -space is étale then it determines a reduced sheaf on G_\bullet .*

Proof. Suppose Z is a G -space and let π_n be the first projection $(G \times Z)_n = G_n \times_{\tilde{p}_n, p} Z \rightarrow G_n$, where $\tilde{p}_n(g_1, \dots, g_n) = s(g_n)$. Then π_n is clearly a simplicial map $(G \times Z)_\bullet \rightarrow G_\bullet$. \square

Corollary 3.8. *Every G -sheaf canonically defines a reduced sheaf over the simplicial space G_\bullet .*

Another example is given by G -modules:

Definition 3.9. Let G be a topological groupoid. A G -module is a topological groupoid A , with source and range maps equal to a map $p: G_0$, such that

- A_x^x is an abelian group for all x ;
- As a space, A is endowed with a G -action $G \times_{s, p} A \rightarrow A$;
- for each $g \in G$, the map $\alpha_g: A_{s(g)} \rightarrow A_{r(g)}$ given by the action is a group morphism.

By Corollary 3.7, any G -module defines a sheaf \mathcal{A}^\bullet which is clearly abelian.

More explicitly, the simplicial structure on $(G \times A)_\bullet$ is defined as follows: for all $f \in \text{Hom}_\Delta(k, n)$,

$$(3.2) \quad \tilde{f}([\gamma_0, \dots, \gamma_n], a) = ([\gamma_{f(0)}, \dots, \gamma_{f(k)}], \gamma_{f(k)}^{-1} \gamma_n a).$$

Then \mathcal{A}^n is the sheaf of germs of continuous sections of $(G \times A)_n \rightarrow G_n$, i.e. sections are continuous maps $\varphi(g_1, \dots, g_n) \in A_{s(g_n)}$. However, to recover the usual formulas like (1.1), it is better to work with the maps

$$c(g_1, \dots, g_n) = g_1 \cdots g_n \varphi(g_1, \dots, g_n) \in A_{r(g_1)}$$

and this is what we shall usually do.

Note that for all $\vec{g}=(g_1, \dots, g_n) \in G_n$, the stalk $\mathcal{A}_{\vec{g}}$ maps to $A_{\tilde{p}_n(\vec{g})}$. This map is surjective iff $p: A \rightarrow G_0$ has enough cross-sections; for injectivity, it is enough that p be an étale map.

If $A=G_0 \times B$ has constant fibers (B being a topological abelian group with no action of G), then the corresponding sheaf is called the constant sheaf and is again (abusively) denoted by B .

When G is a group, a G -module is just a topological abelian group A endowed with a continuous action $G \rightarrow \text{Aut}(A)$ and the sheaf \mathcal{A}^n is just the constant sheaf A on G_n .

3.2. G -sheaves and sheaves over simplicial spaces. Recall (Corollary 3.8) that any G -sheaf defines a sheaf over G_\bullet . In this subsection, which can safely be omitted by the reader, we examine the converse. More precisely:

Proposition 3.10. *Let G be a topological groupoid. There is a one-to-one correspondence between:*

- (i) G -sheaves;
- (ii) reduced sheaves over G_\bullet ;
- (iii) reduced étale spaces over G_\bullet .

Proof. (ii) \iff (iii) was proved in Lemma 3.5 and (i) \implies (ii) is given by Corollary 3.8. For (iii) \implies (i), the proposition below will allow us to conclude. \square

Proposition 3.11. *Let G be a topological groupoid. There is a one-to-one correspondence between reduced morphisms of simplicial spaces $\pi_\bullet: E_\bullet \rightarrow G_\bullet$ and G -spaces. Under this correspondence, étale spaces over G_\bullet are mapped onto G -spaces which are étale over G_0 .*

Proof. The only difficulty is to show that E_0 is endowed with an action of G such that $(G \ltimes E_0)_\bullet \cong E_\bullet$. Consider the map

$$\phi_n: E_n \rightarrow G_n \times E_0^{n+1}$$

defined by $\phi_n=(\pi_n, \tilde{p}_0, \dots, \tilde{p}_n)$ where $p_i: [0] \rightarrow [n]$ is the map $p_i(0)=i$.

One can view $(E_0^{n+1})_{n \in \mathbb{N}}$ as a simplicial space, with

$$\tilde{f}(\xi_0, \dots, \xi_n)=(\xi_{f(0)}, \dots, \xi_{f(n)}) \quad \forall f \in \text{Hom}_\Delta(k, n).$$

It is not hard to show, using the relation $\tilde{p}_j \circ \tilde{f} = \tilde{p}_{f(j)}$ (which holds because $f \circ p_j = p_{f(j)}$), that $\phi_\bullet=(\phi_n)_{n \in \mathbb{N}}$ is a morphism of simplicial spaces.

Since E_\bullet is reduced, (π_n, \tilde{p}_n) is a homeomorphism from E_n to $G_n \times_{\tilde{p}_n, \pi_0} E_0$ and in particular ϕ_n is injective.

Define the action of G on E_0 be the composition of maps

$$G \times_{s,p} E_0 \xrightarrow{(\pi_1, \tilde{p}_1)^{-1}} E_1 \xrightarrow{\tilde{p}_0} E_0,$$

i.e.

$$(3.3) \quad \xi' = g\xi \iff \exists x \in E_1, \phi_1(x)=(g, \xi', \xi).$$

Let $x \in E_n$, let $(g_1, \dots, g_n) = \pi_n(x)$ and $\xi = \tilde{p}_n(x)$. Let us show that

$$(3.4) \quad \phi_n(x) = (\pi_n(x), g_1 \cdots g_n \xi, g_2 \cdots g_n \xi, \dots, \xi),$$

where $g_1 \cdots g_n \xi = g_1(g_2(\cdots(g_n \xi) \cdots))$ (we don't know yet that $(g, \xi) \mapsto g\xi$ is an action).

Consider the map $f: [1] \rightarrow [n]$ such that $f(0) = j - 1$ and $f(1) = j$. Let $y = \tilde{f}(x) \in E_1$, then

$$\phi_1(y) = \phi_1(\tilde{f}(x)) = \tilde{f}(\phi_n(x)) = (g_j, \tilde{p}_{j-1}(x), \tilde{p}_j(x)),$$

hence by (3.3), $\tilde{p}_{j-1}(x) = g_j \tilde{p}_j(x)$. Equation (3.4) follows by reverse induction on j .

Now, let $(g, h, \xi) \in G_2 \times_{\tilde{p}_2, p} E_0$. We want to show that $g(h\xi) = (gh)\xi$. Let $x = (\pi_2, \tilde{p}_2)^{-1}(g, h, \xi) \in E_2$. By Equation (3.4) above, $\phi_2(x) = (g, h, g(hx), hx, x)$. It follows that $\phi_1(\tilde{\epsilon}_1(x)) = \tilde{\epsilon}_1(\phi_2(x)) = (gh, g(hx), x)$. Using again (3.3), we get $(gh)x = g(hx)$.

To show that $p(\xi)\xi = \xi$ for all $\xi \in E_0$, let $x = \tilde{\eta}_0(\xi) \in E_1$, then

$$\phi_1(x) = (\pi_1(x), \tilde{p}_0(x), \tilde{p}_1(x)) = (p(\xi), \xi, \xi),$$

therefore, by (3.3), we get $p(\xi)\xi = \xi$.

We have shown that G acts on E_0 . To show that the simplicial spaces $(G \times E_0)_\bullet$ and E_\bullet are isomorphic, let $\phi'_n: G_n \times_{\tilde{p}_n, p} E_0 = (G \times E_0)_n \rightarrow G_n \times E_0^{n+1}$ be defined like ϕ , then ϕ'_n is also injective for all n and ϕ'_\bullet is a morphism of simplicial spaces. By Equation (3.4), we have a commutative diagram

$$\begin{array}{ccc} E_n & \xrightarrow{(\pi_n, \tilde{p}_n)} & G_n \times_{\tilde{p}_n, p} E_0 \\ & \searrow \phi_n & \downarrow \phi'_n \\ & & G_n \times E_0^{n+1} \end{array}$$

where ϕ_n and ϕ'_n are injective and (π_n, \tilde{p}_n) is a homeomorphism. Therefore, $(\pi_n, \tilde{p}_n)_{n \in \mathbb{N}}$ is an isomorphism of simplicial spaces. \square

4. ČECH COHOMOLOGY

4.1. Covers of simplicial spaces.

Definition 4.1. An *open cover* of a semi-simplicial space M_\bullet is a sequence $\mathcal{U}_\bullet = (\mathcal{U}_n)_{n \in \mathbb{N}}$ such that $\mathcal{U}_n = (U_i^n)_{i \in I_n}$ is an open cover of the space M_n .

The cover is said to be *semi-simplicial* if $I_\bullet = (I_n)_{n \in \mathbb{N}}$ is a semi-simplicial set such that for all $f \in \text{Hom}_\Delta(k, n)$ and for all $i \in I_n$ one has $\tilde{f}(U_i^n) = U_{\tilde{f}(i)}^k$. In the same way, one defines the notions of simplicial cover and of N -simplicial cover.

The reason why we need to introduce this terminology is that, even when M_\bullet is a simplicial space, there may not exist sufficiently fine simplicial covers. However, given a cover \mathcal{U}_\bullet , we can form the semi-simplicial cover $\sigma\mathcal{U}_\bullet$ defined as follows.

Let $\mathcal{P}_n = \cup_{k=0}^n \mathcal{P}_n^k$, where $\mathcal{P}_n^k = \text{Hom}_{\Delta'}(k, n)$. Note that \mathcal{P}_n can be identified with the set of nonempty subsets of $[n]$.

Let Λ_n (or $\Lambda_n(I)$ if there is a risk of confusion) be the set of maps

$$(4.1) \quad \lambda: \mathcal{P} \rightarrow \cup_k I_k \text{ such that } \lambda(\mathcal{P}_n^k) \subset I_k.$$

For all $\lambda \in \Lambda_n$, we let

$$U_\lambda^n = \bigcap_{k \leq n} \bigcap_{f \in \mathcal{P}_n^k} \tilde{f}^{-1}(U_{\lambda(f)}^k).$$

It is clear that $(U_\lambda)_{\lambda \in \Lambda_n}$ is an open cover of M_n .

The semi-simplicial structure on Λ_\bullet is defined in an obvious way: for all $g \in \text{Hom}_{\Delta'}(n, n')$, $\tilde{g}: \Lambda_{n'} \rightarrow \Lambda_n$ is the map

$$(\tilde{g}\lambda')(f) = \lambda'(g \circ f).$$

In the same way, for all integers $n \leq N$, let

$$(4.2) \quad (\sigma_N \mathcal{U})_n = (U_\lambda^n)_{\lambda \in \Lambda_n^N},$$

where Λ_n^N is the set of all maps $\lambda: \bigcup_{k \leq n} \text{Hom}_{\Delta}(k, n) \rightarrow \bigcup_{k \leq n} I_k$ which satisfy $\lambda(\text{Hom}_{\Delta}(k, n)) \subset I_k$, and

$$U_\lambda^n = \bigcap_{k \leq n} \bigcap_{f \in \text{Hom}_{\Delta}(k, n)} \tilde{f}^{-1}(U_\lambda^k(f)).$$

The N -simplicial structure on Λ_\bullet^N is defined as follows: for all integers $n, n' \leq N$ and all $g \in \text{Hom}_{\Delta}(n, n')$, $\tilde{g}: \Lambda_{n'}^N \rightarrow \Lambda_n^N$ is the map $(\tilde{g}\lambda')(f) = \lambda'(g \circ f)$.

Then $\sigma_N \mathcal{U}_\bullet = (\sigma_N \mathcal{U}_n)_{n \leq N}$ is a N -simplicial cover of the N -skeleton of M_\bullet .

Convention 4.2. We will also (abusively) denote by $\sigma_N \mathcal{U}_\bullet$ the open cover which coincides with $\sigma_N \mathcal{U}_\bullet$ for $n \leq N$ and with \mathcal{U}_\bullet for $n \geq N + 1$.

Example 4.3. Let $M_\bullet = (M)_{n \in \mathbb{N}}$ be the constant simplicial space associated to a topological space M , and suppose $\mathcal{U}_0 = (U_i^0)_{i \in I_0}$ is an open cover of M . Define $I_n = I_0^{n+1}$, then $I_\bullet = (I_n)_{n \in \mathbb{N}}$ is endowed with a simplicial structure by $\tilde{f}(i_0, \dots, i_n) = (i_{f(0)}, \dots, i_{f(n)})$ for all $f \in \text{Hom}_{\Delta}(k, n)$. Let $U_{(i_0, \dots, i_n)}^n = U_{i_0}^0 \cap \dots \cap U_{i_n}^0$, and let $\mathcal{U}_n = (U_i^n)_{i \in I_n}$, then \mathcal{U}_\bullet is a simplicial cover of M_\bullet .

The “set” of covers of a simplicial space M_\bullet is endowed with a partial preorder. Suppose \mathcal{U}_\bullet and \mathcal{V}_\bullet are open covers of M_\bullet , with $\mathcal{U}_n = (U_i^n)_{i \in I_n}$ and $\mathcal{V}_n = (V_j^n)_{j \in J_n}$. We say that \mathcal{V} is *finer* than \mathcal{U} if for all n there exists $\theta_n: J_n \rightarrow I_n$ such that $\theta_n(V_j^n) \subset U_{\theta_n(j)}^n$ for all j . The map $\theta_\bullet = (\theta_n)_{n \in \mathbb{N}}$ is required to be semi-simplicial (resp. N -simplicial) if \mathcal{U} and \mathcal{V} are semi-simplicial (resp. N -simplicial).

4.2. Čech cohomology. Let \mathcal{U}_\bullet be a semi-simplicial open cover of M_\bullet and let \mathcal{A}^\bullet be a semi-simplicial abelian sheaf. Define a complex

$$C_{ss}^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) = \prod_{i \in I_n} \mathcal{A}^n(U_i^n),$$

i.e. $C_{ss}^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ is the space of global sections of the pull-back of \mathcal{A}^n on $\prod_{i \in I_n} U_i^n$.

Define the differential $d: C_{ss}^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C_{ss}^{n+1}(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ by

$$(dc)_i = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k^* c_{\tilde{\varepsilon}_k(i)},$$

where $\tilde{\varepsilon}_k^* c_{\tilde{\varepsilon}_k(i)}$ is the “restriction” of $c_{\tilde{\varepsilon}_k(i)} \in \mathcal{A}^n(U_{\tilde{\varepsilon}_k(i)}^n)$ to a section in $\mathcal{A}^{n+1}(U_i^{n+1})$.

It is immediate to check that $d^2=0$, hence we may define the cohomology groups $H_{ss}^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$.

Example 4.4. In Example 4.3, suppose \mathcal{A} is an abelian sheaf on M and that $\mathcal{A}^n = \mathcal{A}$ for all n . Then $H_{ss}^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ is identical to the usual cohomology group $H^*(\mathcal{U}_0; \mathcal{A})$.

Let \mathcal{U}_\bullet be any open cover of M_\bullet . We denote

$$(4.3) \quad C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) = C_{ss}^n(\sigma\mathcal{U}_\bullet; \mathcal{A}^\bullet)$$

$$(4.4) \quad H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) = H_{ss}^n(\sigma\mathcal{U}_\bullet; \mathcal{A}^\bullet).$$

Now, we want to define Čech cohomology. The idea is to define $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ as the inductive limit over \mathcal{U}_\bullet of the groups $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$. The problem is that if $\theta_\bullet: J_\bullet \rightarrow I_\bullet$ is a refinement, then θ_\bullet indeed defines a restriction map

$$(4.5) \quad \theta^*: C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C^n(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$$

$$(4.6) \quad (\theta^* \varphi)_j = \text{restriction to } V_j^n \text{ of } \varphi_{\theta_n(j)}$$

which commutes with the differentials, and thus θ^* defines a map

$$\theta^*: H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow H^n(\mathcal{V}_\bullet; \mathcal{A}^\bullet).$$

However, that map may depend on the choice of θ . On the other hand we have the

Lemma 4.5. *Let $N \in \mathbb{N}$. Suppose that \mathcal{U}_\bullet and \mathcal{V}_\bullet are open covers of M_\bullet such that \mathcal{V}_\bullet admits a N -simplicial structure. Suppose that \mathcal{V}_\bullet is finer than \mathcal{U}_\bullet and that $\theta_0, \theta_1: \mathcal{U}_\bullet \rightarrow \mathcal{V}_\bullet$ are two refinements. Then for all $n \leq N$ there exists $H: C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C^{n-1}(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$ (with the convention $C^{-1} = \{0\}$) such that $\theta_1^* - \theta_0^* = dH + Hd$.*

(In the lemma above, we say that \mathcal{V}_\bullet has a N -simplicial structure if the N -skeleton $(\mathcal{V}_n)_{n \in \mathbb{N}}$ has a N -simplicial structure.)

Proof. Define for all $\varphi \in C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ and for all $\lambda \in \Lambda_{n-1}(J)$ (recall notation (4.1)):

$$(4.7) \quad (H\varphi)_\lambda = \sum_{k=0}^{n-1} (-1)^k \tilde{\eta}_k^* \varphi_{\alpha_k(\lambda)},$$

where as usual $\eta_k: [n] \rightarrow [n-1]$ is the k -th degeneracy map, and α_k is defined as follows: for all $f \in \text{Hom}_{\Delta'}(r, n)$, let

$$\alpha_k(\lambda)(f) = \begin{cases} \theta_0(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \leq k \\ \theta_1(\lambda(\eta_k \circ f)) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \geq k+1 \\ \theta_0(\tilde{\eta}_{k'}(\lambda(f'))) & \text{if } \{k, k+1\} \subset f([r]) \end{cases}$$

where in the third line, k' is the integer such that $f(k')=k$ and f' is the unique morphism in $\text{Hom}_{\Delta'}(r-1, n-1)$ such that the following diagram commutes:

$$(4.8) \quad \begin{array}{ccc} [r] & \xrightarrow{f} & [n] \\ \downarrow \eta_{k'} & & \downarrow \eta_k \\ [r-1] & \xrightarrow{f'} & [n-1] \end{array}$$

i.e. $f'(i)=f(i)$ for $i \leq k'$ and $f'(i)=f(i+1)-1$ for $i \geq k'+1$.

Let us first check that formula (4.7) makes sense, i.e. that $V_\lambda^{n-1} \subset \tilde{\eta}_k^{-1}(U_{\alpha_k(\lambda)}^r)$ for all k .

Since $U_{\alpha_k(\lambda)}^n = \bigcap_{r=0}^n \bigcap_{f \in \text{Hom}_{\Delta'}(r, n)} \tilde{f}^{-1}(U_{\alpha_k(\lambda)(f)}^n)$, we need to show that

$$(4.9) \quad V_\lambda^{n-1} \subset \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U_{\alpha_k(\lambda)(f)}^r).$$

If $\{k, k+1\} \not\subset f([r])$ and $f(0) \leq k$ then

$$\begin{aligned} V_\lambda^{n-1} &\subset \widetilde{(\eta_k \circ f)}^{-1}(V_{\lambda(\eta_k \circ f)}^r) \quad \text{by definition of } V_\lambda^{n-1} \\ &= \tilde{\eta}_k^{-1} \tilde{f}^{-1}(V_{\lambda(\eta_k \circ f)}^r) \\ &\subset \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U_{\theta_0(\lambda(\eta_k \circ f))}^r) \quad \text{since } \theta_0: J_\bullet \rightarrow I_\bullet \text{ is a refinement} \\ &= \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U_{\alpha_k(\lambda)(f)}^r). \end{aligned}$$

If $\{k, k+1\} \not\subset f([r])$ and $f(0) \geq k+1$ the proof of (4.9) is the same, except that θ_0 is replaced by θ_1 .

If $\{k, k+1\} \subset f([r])$ then

$$\begin{aligned} V_\lambda^{n-1} &\subset (\tilde{f}')^{-1}(V_{\lambda(f')}^{r-1}) \\ &\subset \tilde{f}'^{-1} \tilde{\eta}_{k'}^{-1}(V_{\tilde{\eta}_k(\lambda(f'))}^r) \quad (\text{recall } \mathcal{V}_\bullet \text{ is } N\text{-simplicial}) \\ &\subset \tilde{\eta}_k^{-1} \tilde{f}^{-1}(U_{\theta_0(\tilde{\eta}_k(\lambda(f')))}^r) \quad \text{by (4.8),} \end{aligned}$$

thus (4.9) is proved.

Let us show that

$$(4.10) \quad dH + Hd = \theta_1^* - \theta_0^*.$$

We have

$$(Hd\varphi)_\lambda = \sum_{\ell=0}^n \sum_{k=0}^{n+1} A_{k,\ell} \quad \text{and} \quad (dH\varphi)_\lambda = \sum_{k=0}^n \sum_{\ell=0}^{n-1} B_{k,\ell},$$

where $A_{k,\ell} = (-1)^{k+\ell} \tilde{\eta}_\ell^* \tilde{\varepsilon}_k^* \varphi_{\tilde{\varepsilon}_k(\alpha_\ell(\lambda))}$ and $B_{k,\ell} = (-1)^{k+\ell} \tilde{\varepsilon}_k^* \tilde{\eta}_\ell^* \varphi_{\alpha_\ell(\tilde{\varepsilon}_k(\lambda))}$.

We have $A_{\ell,\ell} = \tilde{\eta}_\ell^* \tilde{\varepsilon}_\ell^* \varphi_{\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))} = \varphi_{\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))}$, and for all $f \in \mathcal{P}_n$,

$$\tilde{\varepsilon}_\ell(\alpha_\ell(\lambda))(f) = \alpha_\ell(\lambda)(\varepsilon_\ell \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_\ell(f))) = \theta_j(\lambda(f)),$$

where $j=0 \iff \varepsilon_\ell \circ f(0) \leq \ell \iff f(0) \leq \ell-1$, and $j=1$ otherwise.

$$\text{Let } \lambda^{(p)}(f) = \begin{cases} \theta_0(\lambda(f)) & \text{if } f(0) \leq p \\ \theta_1(\lambda(f)) & \text{if } f(0) \geq p+1, \end{cases} \quad \text{then}$$

$$(4.11) \quad A_{\ell,\ell} = \varphi_{\lambda^{\ell-1}}.$$

Similarly, we have $A_{\ell+1,\ell} = -\tilde{\eta}_\ell^* \tilde{\varepsilon}_{\ell+1}^* \varphi_{\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))} = -\varphi_{\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))}$, and for all $f \in \mathcal{P}_n$,

$$\tilde{\varepsilon}_{\ell+1}(\alpha_\ell(\lambda))(f) = \alpha_\ell(\lambda)(\varepsilon_{\ell+1} \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_{\ell+1} \circ f)) = \theta_j(\lambda(f)),$$

where $j=0 \iff \varepsilon_{\ell+1} \circ f(0) \leq \ell \iff f(0) \leq \ell$, and $j=1$ otherwise. We thus get

$$(4.12) \quad A_{\ell+1,\ell} = -\varphi_{\lambda^\ell}.$$

From (4.11) and (4.12) we obtain $\sum_{\ell \leq k \leq \ell+1} A_{k,\ell} = \varphi_{\lambda^{(-1)}} - \varphi_{\lambda^{(n)}} = \theta_1^* \varphi - \theta_0^* \varphi$.

Let us examine the other terms. We have $\sum_{k \leq \ell-1} A_{k,\ell} = \sum_{0 \leq k \leq \ell \leq n-1} A_{k,\ell+1}$ and $\sum_{k \geq \ell+2} A_{k,\ell} = \sum_{0 \leq \ell < k \leq n} A_{k+1,\ell}$. To complete the proof of (4.10) it suffices to show that $A_{k,\ell+1} + B_{k,\ell} = 0$ for $k \leq \ell$, and that $A_{k+1,\ell} + B_{k,\ell} = 0$ for $k \geq \ell+1$. Noting that $\eta_{\ell+1} \varepsilon_k = \varepsilon_k \eta_\ell$ for $k \leq \ell$ and that $\eta_\ell \varepsilon_{k+1} = \varepsilon_k \eta_\ell$ for $k \geq \ell+1$, it suffices to show that

- (a) $\tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))$ for $k \leq \ell$,
- (b) $\tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))$ for $k \geq \ell+1$.

Let us show (a). Suppose that $f \in \text{Hom}_{\Delta'}(r, n)$ and let us first treat the case $\{\ell, \ell+1\} \not\subset f([r])$. Then, letting $j=0$ for $\varepsilon_k \circ f(0) \leq \ell+1$ ($\iff f(0) \leq \ell$), and $j=1$ otherwise, we have

$$\begin{aligned} \tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda))(f) &= \alpha_{\ell+1}(\lambda)(\varepsilon_k \circ f) = \theta_j(\lambda(\eta_{\ell+1} \circ \varepsilon_k \circ f)) \\ &= \theta_j(\lambda(\varepsilon_k \circ \eta_\ell \circ f)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))(f). \end{aligned}$$

Let us treat the case $\{\ell, \ell+1\} \subset f([r])$. Let ℓ' such that $f(\ell') = \ell$, and let $f': [r-1] \rightarrow [n-1]$ be the increasing map such that the diagram

$$(4.13) \quad \begin{array}{ccc} [r] & \xrightarrow{f} & [n] \\ \downarrow \eta_{\ell'} & & \downarrow \eta_\ell \\ [r-1] & \xrightarrow{f'} & [n-1] \end{array}$$

commutes. Since $\varepsilon_k \circ \eta_\ell = \eta_{\ell+1} \circ \varepsilon_k : [n] \rightarrow [n]$, the diagram

$$(4.14) \quad \begin{array}{ccc} [r] & \xrightarrow{\varepsilon_k \circ f} & [n+1] \\ \downarrow \eta_{\ell'} & & \downarrow \eta_{\ell+1} \\ [r-1] & \xrightarrow{\varepsilon_k \circ f'} & [n] \end{array}$$

commutes. We thus see that

$$\begin{aligned} \tilde{\varepsilon}_k(\alpha_{\ell+1}(\lambda))(f) &= \alpha_{\ell+1}(\lambda)(\varepsilon_k \circ f) \\ &= \theta_0(\tilde{\eta}_{\ell'}(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.14)} \\ &= \theta_0(\tilde{\eta}_{\ell'}((\tilde{\varepsilon}_k \lambda)(f'))) \\ &= \alpha_\ell(\tilde{\varepsilon}_k \lambda)(f) \quad \text{by (4.13)}. \end{aligned}$$

This completes the proof of (a).

Let us show (b): the method is similar. If $\{\ell, \ell+1\} \not\subset f([r])$ then

$$\begin{aligned} \tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda))(f) &= \alpha_\ell(\lambda)(\varepsilon_{k+1} \circ f) = \theta_j(\lambda(\eta_\ell \circ \varepsilon_{k+1} \circ f)) \\ &= \theta_j(\lambda(\varepsilon_k \circ \eta_\ell \circ f)) = \theta_j((\tilde{\varepsilon}_k \lambda)(\eta_\ell \circ f)) = \alpha_\ell(\tilde{\varepsilon}_k(\lambda))(f). \end{aligned}$$

If $\{\ell, \ell+1\} \subset f([r])$, let ℓ' such that $f(\ell') = \ell$ and let f' such that

$$(4.15) \quad \begin{array}{ccc} [r] & \xrightarrow{f} & [n] \\ \downarrow \eta_{\ell'} & & \downarrow \eta_\ell \\ [r-1] & \xrightarrow{f'} & [n-1] \end{array}$$

commutes. Then since $\varepsilon_k \circ \eta_\ell = \eta_\ell \circ \varepsilon_{k+1} : [n] \rightarrow [n]$, the diagram

$$(4.16) \quad \begin{array}{ccc} [r] & \xrightarrow{\varepsilon_{k+1} \circ f} & [n+1] \\ \downarrow \eta_{\ell'} & & \downarrow \eta_\ell \\ [r-1] & \xrightarrow{\varepsilon_k \circ f'} & [n] \end{array}$$

commutes, therefore

$$\begin{aligned} \tilde{\varepsilon}_{k+1}(\alpha_\ell(\lambda))(f) &= \alpha_\ell(\lambda)(\varepsilon_{k+1} \circ f) \\ &= \theta_0(\tilde{\eta}_{\ell'}(\lambda(\varepsilon_k \circ f'))) \quad \text{by (4.16)} \\ &= \theta_0(\tilde{\eta}_{\ell'}((\tilde{\varepsilon}_k \lambda)(f'))) \\ &= (\alpha_\ell(\tilde{\varepsilon}_k \lambda))(f) \quad \text{by (4.15)}. \end{aligned}$$

This completes the proof of (b) and hence of (4.10). \square

Let us now define

$$(4.17) \quad \check{H}^n(M_\bullet; \mathcal{A}^\bullet) = \lim_{\rightarrow} H^n(\mathcal{U}_\bullet; A^\bullet)$$

where \mathcal{U}_\bullet runs over open covers of M_\bullet whose N -skeleton admits a N -simplicial structure for some $N \geq n+1$. (Recall that $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ was defined by equation (4.4).)

To avoid set-theoretic difficulties (since the collection of open covers is not a set), we can restrict ourselves to open covers indexed by sets of cardinality $\leq \sum_n \#M_n$.

By Lemma 4.5 above, if \mathcal{V}_\bullet is finer than \mathcal{U}_\bullet and if \mathcal{V}_\bullet has a N -simplicial structure, then there is a canonical map $H^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow H^*(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$ defined by Eqn. (4.6) (θ is *not* required to respect the N -simplicial structures). Since every open cover of M_\bullet admits a N -simplicial refinement (see Convention 4.2), the inductive limit is well-defined and is an abelian group.

Moreover, for every open cover \mathcal{U}_\bullet of M_\bullet , N -simplicial or not, there is a canonical map $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow \check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ obtained by mapping $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ to $H^n(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$ using (4.6), where \mathcal{V}_\bullet is any refinement admitting a N -simplicial structure for some $N \geq n+1$.

It is clear that any element $[\varphi]$ of $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ maps to 0 in $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ if and only if there exists a refinement (N -simplicial or not) such that $[\varphi]$ maps to 0 in $H^n(\mathcal{V}_\bullet; \mathcal{A}^\bullet)$. Thus, in some sense, we can say that $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ is the inductive limit of $H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ where \mathcal{U}_\bullet runs over *all* open covers of M_\bullet .

Example 4.6. Consider the case of a discrete group G , and suppose that \mathcal{A}^\bullet is the sheaf associated to a G -module A (Definition 3.9). Then, from (3.2) and below, we see that

$$\begin{aligned} (dc)_\lambda(g_1, \dots, g_{n+1}) &= g_1 c_{\tilde{\varepsilon}_0 \lambda}(g_2, \dots, g_{n+1}) + \\ &+ \sum_{k=1}^n (-1)^k c_{\tilde{\varepsilon}_k \lambda}(g_1, \dots, g_k g_{k+1}, \dots, g_{n+1}) + \\ &+ (-1)^{n+1} c_{\tilde{\varepsilon}_{n+1} \lambda}(g_1, \dots, g_n). \end{aligned}$$

(Compare with (1.1).) Considering the maximal open cover $(U_x^n)_{x \in G^n}$ where $U_x^n = \{x\}$, one easily sees that Čech cohomology coincides with usual group cohomology.

Remark 4.7. As in [5], $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ can be seen as the n -th cohomology group of a canonical Čech complex. Indeed, let $\mathcal{R}(M_\bullet)$ be the set of covers of the form $\mathcal{U}_n = (U_x^n)_{x \in M_n}$. If \mathcal{U}_\bullet and \mathcal{V}_\bullet are in $\mathcal{R}(M_\bullet)$, let us say that \mathcal{V}_\bullet is finer than \mathcal{U}_\bullet if for all n and all $x \in M_n$ we have $V_x^n \subset U_x^n$. Given \mathcal{U}_\bullet in $\mathcal{R}(M_\bullet)$, denote by $\sigma_N \mathcal{U}_\bullet$ the associated N -simplicial cover (see Convention 4.2) and let

$$\check{C}_N^m(M_\bullet; \mathcal{A}^\bullet) = \lim_{\mathcal{U}_\bullet} C^m(\sigma_N \mathcal{U}_\bullet; \mathcal{A}^\bullet) = \lim_{\mathcal{U}_\bullet} C_{ss}^m(\sigma \sigma_N \mathcal{U}_\bullet; \mathcal{A}^\bullet),$$

where \mathcal{U}_\bullet runs over open covers in $\mathcal{R}(M_\bullet)$.

Then $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ is the cohomology of $\check{C}_N^m(M_\bullet; \mathcal{A}^\bullet)$ whenever $N \geq n+1$.

4.3. Compatibility with usual Čech cohomology for spaces. Let M be a space and \mathcal{A} an abelian sheaf on M . Denote by M_\bullet be the constant simplicial space associated to M and by \mathcal{A}^\bullet the sheaf on M_\bullet corresponding to \mathcal{A} .

We want to show the

Proposition 4.8. *With the above assumptions, the usual Čech cohomology groups $\check{H}^n(M; \mathcal{A})$ are isomorphic to the Čech cohomology groups $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$.*

Proof. To determine $\check{H}^n(M_\bullet; \mathcal{A}^\bullet)$ we can restrict ourselves to covers \mathcal{U}_\bullet of the form $\mathcal{U}_n = (U_i^n)_{i \in I_n}$ where $I_n = I_0^{n+1}$ and $U_{i_0, \dots, i_n}^n = U_{i_0}^0 \cap \dots \cap U_{i_n}^0$. Let us show that $H^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong H^*(\mathcal{U}_0; \mathcal{A})$. It is not obvious that these two groups are isomorphic, since $C^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet) = C_{ss}^*(\sigma \mathcal{U}_\bullet; \mathcal{A}^\bullet)$ while $C^*(\mathcal{U}_0; \mathcal{A}) = C_{ss}^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$. However, we show that these two complexes are homotopically equivalent:

First, there is an obvious map

$$q: C^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C^*(\mathcal{U}_0; \mathcal{A})$$

defined by $(q\varphi)_{i_0, \dots, i_n} = \varphi_{\lambda(i)}$, where

$$\lambda^{(i)}(f) = (i_{f(0)}, \dots, i_{f(r)}) \quad \text{for all } f \in \text{Hom}_{\Delta'}(r, n).$$

In the other direction, define $\iota: C^*(\mathcal{U}_0; \mathcal{A}) \rightarrow C^*(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ by

$$(\iota c)_\lambda = c_{\lambda_0, \dots, \lambda_n},$$

where λ_k denotes $\lambda(p_k)$ and $p_k: [0] \rightarrow [n]$ denotes the map $p_k(0) = k$.

We have $q \circ \iota = \text{Id}$. Indeed, $((q \circ \iota)(c))_{i_0, \dots, i_n} = (\iota c)_{\lambda(i)} = c_{i_0, \dots, i_n}$.

Conversely, we don't have $\iota \circ q = \text{Id}$ since $((\iota \circ q)(\varphi))_\lambda = \varphi_{\lambda'}$, where

$$\lambda'(f) = (\lambda_{f(0)}, \dots, \lambda_{f(r)}) \quad \text{for all } f \in \text{Hom}_{\Delta'}(r, n).$$

However, $\iota \circ q$ and Id are homotopic. Indeed, define $H: C^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C^{n-1}(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ by

$$(4.18) \quad (H\varphi)_\lambda = \sum_{k=0}^{n-1} (-1)^k \tilde{\eta}_k^* \varphi_{\alpha_k(\lambda)}$$

where

$$\alpha_k(\lambda)(f) = \begin{cases} \lambda(\eta_k \circ f) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \leq k \\ \lambda(\eta_k \circ f) & \text{if } \{k, k+1\} \not\subset f([r]) \text{ and } f(0) \geq k+1 \\ \tilde{\eta}_{k'}(\lambda(f')) & \text{if } \{k, k+1\} \subset f([r]) \end{cases}$$

$(f(k')) = k$ and f' is defined as in the proof of Lemma 4.5; also, recall that $\tilde{\eta}_{k'}(i_0, \dots, i_{r-1}) = (i_0, \dots, i_{k'}, i_{k'}, \dots, i_{r-1})$.

Then the same proof as in Lemma 4.5 shows that $dH + Hd = \iota \circ q - \text{Id}$. We leave out details; anyway we will show later that Čech cohomology coincides with sheaf cohomology for paracompact simplicial spaces, so (at least in the paracompact case) this will provide a second proof that sheaf cohomology for spaces coincides with sheaf cohomology for the associated constant simplicial space. \square

Let us introduce some notation:

Notation 4.9. For any simplicial space M_\bullet and any open cover \mathcal{U}_\bullet , let us write elements $\lambda \in \Lambda_n$ (see (4.1)) as $(2^{n+1} - 1)$ -tuples $(\lambda_S)_{\emptyset \neq S \subset [n]}$, where subsets S are ordered first by cardinality, then by lexicographic order. For instance, the triple $(\lambda_0, \lambda_1, \lambda_{01})$ represents the element $\lambda \in \Lambda_1$ such that $\lambda(\{0\}) = \lambda_0$, $\lambda(\{1\}) = \lambda_1$, $\lambda(\{0, 1\}) = \lambda_{01}$. A cochain in $C^1(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ is thus a family $(\varphi_{\lambda_0 \lambda_1 \lambda_{01}})$.

Then, we can write (4.18) more explicitly. For instance, the formulas for $n=1$ and $n=2$ are respectively

$$\begin{aligned} (H\varphi)_{\lambda_0} &= \varphi_{\lambda_0 \lambda_0 \lambda'_{00}} \\ (H\varphi)_{\lambda_0 \lambda_1 \lambda_{01}} &= \varphi_{\lambda_0 \lambda_0 \lambda_1 \lambda'_{00} \lambda_{01} \lambda'_{01} \lambda'_{001}} - \varphi_{\lambda_0 \lambda_1 \lambda_1 \lambda_{01} \lambda_{01} \lambda'_{11} \lambda'_{011}}, \end{aligned}$$

where $\lambda'_{i_0 \dots i_r}$ denotes the $r+1$ -tuple $(\lambda_{i_0}, \dots, \lambda_{i_r}) \in I_0^{r+1} = I_r$.

4.4. Long exact sequences in Čech cohomology. In this section, most proofs are almost identical to [5], thus we will only sketch them.

Proposition 4.10. *If $0 \rightarrow \mathcal{A}^\bullet \rightarrow \mathcal{A}^\bullet \rightarrow \mathcal{A}''^\bullet \rightarrow 0$ is an exact sequence of abelian presheaves then the functor $\mathcal{A} \mapsto \check{C}_N^*(M_\bullet; \mathcal{A}^\bullet)$ (see Remark 4.7) maps the above exact sequence to an exact sequence of complexes.*

Proof. $0 \rightarrow C_{ss}^*(\sigma\sigma_N \mathcal{U}_\bullet; \mathcal{A}''^\bullet) \rightarrow C_{ss}^*(\sigma\sigma_N \mathcal{U}_\bullet; \mathcal{A}^\bullet) \rightarrow C_{ss}^*(\sigma\sigma_N \mathcal{U}_\bullet; \mathcal{A}''^\bullet) \rightarrow 0$ is exact for every open cover \mathcal{U}_\bullet . \square

Let us say that a simplicial space M_\bullet is *paracompact* if each M_n is paracompact.

Proposition 4.11. [5, theorem 5.10.2] *If \mathcal{A}^\bullet is an abelian presheaf on a paracompact simplicial space M_\bullet such that \mathcal{A}^\bullet induces the zero sheaf then $\check{H}^n(M_\bullet; \mathcal{A}^\bullet) = 0$ for all $n \geq 0$.*

Proof. Using paracompactness, every cohomology class is represented by a cocycle in $C_N^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ with \mathcal{U}_n locally finite $\forall n$. Then, using the fact that each \mathcal{A}^n induces the zero sheaf, every cochain of that cover becomes zero once restricted to a suitable finer cover. \square

Corollary 4.12. *If an abelian presheaf \mathcal{A}^\bullet over a paracompact simplicial space M_\bullet induces the sheaf $\tilde{\mathcal{A}}^\bullet$ then $\check{H}^n(M_\bullet; \mathcal{A}^\bullet) \cong \check{H}^n(M_\bullet; \tilde{\mathcal{A}}^\bullet)$.*

Proof. There are exact sequences of presheaves

$$\begin{aligned} 0 \rightarrow \mathcal{N}^\bullet \rightarrow \mathcal{A}^\bullet \rightarrow \mathcal{J}^\bullet \rightarrow 0 \\ 0 \rightarrow \mathcal{J}^\bullet \rightarrow \tilde{\mathcal{A}}^\bullet \rightarrow \mathcal{Q}^\bullet \rightarrow 0, \end{aligned}$$

where \mathcal{N}^\bullet and \mathcal{Q}^\bullet induce the zero sheaf: $\mathcal{N}^n(U)$ is the set of sections in $\mathcal{A}^n(U)$ whose germ at every point is zero, and $\tilde{\mathcal{A}}^n(U) = \{(\sigma_i)_{i \in I}\} / \sim$, where $\sigma_i \in \mathcal{J}^n(U_i)$ for some open cover $(U_i)_{i \in I}$ of U and the equivalence relation

\sim is defined by $(\sigma_i)_{i \in I} \sim (\sigma'_j)_{j \in J}$ iff $\forall i, j, \sigma_i|_{U_i \cap U'_j} = \sigma'_j|_{U_i \cap U'_j}$. The conclusion follows from Propositions 4.10 and 4.11 above. \square

Corollary 4.13. *If $0 \rightarrow \mathcal{A}'^\bullet \rightarrow \mathcal{A}^\bullet \rightarrow \mathcal{A}''^\bullet$ is an exact sequence of sheaves over a paracompact simplicial space M_\bullet , then there is a natural long exact sequence*

$$0 \rightarrow \check{H}^0(M_\bullet; \mathcal{A}'^\bullet) \rightarrow \check{H}^0(M_\bullet; \mathcal{A}^\bullet) \rightarrow \check{H}^0(M_\bullet; \mathcal{A}''^\bullet) \xrightarrow{\partial} \check{H}^1(M_\bullet; \mathcal{A}'^\bullet) \rightarrow \dots$$

Proof. Follows from Proposition 4.10 and Corollary 4.12. \square

5. LOW DIMENSIONAL ČECH COHOMOLOGY

5.1. The group \check{H}^0 . Consider a simplicial space M_\bullet . Let \mathcal{U}_\bullet be an open cover of M_\bullet , then, using Notation 4.9, a 0-cocycle is given by a family $(c_{\lambda_0})_{\lambda_0 \in I_0}$, with $c_{\lambda_0} \in \mathcal{A}^0(U_{\lambda_0}^0)$, and

$$(5.1) \quad 0 = (dc)_{\lambda_0 \lambda_1 \lambda_{01}} = \tilde{\varepsilon}_1^* c_{\lambda_1} - \tilde{\varepsilon}_0^* c_{\lambda_0}$$

on $U_\lambda^1 = U_{\lambda_{01}}^1 \cap \tilde{\varepsilon}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_1}^0)$. Therefore, $\tilde{\varepsilon}_1^* c_{\lambda_1} = \tilde{\varepsilon}_0^* c_{\lambda_0}$ on $\tilde{\varepsilon}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_1}^0)$ for all $\lambda_0, \lambda_1 \in I_0$. Applying $\tilde{\eta}_0^*$ to both sides, we find that $c_{\lambda_0} = c_{\lambda_1}$ on $U_{\lambda_0}^0 \cap U_{\lambda_1}^0$. Since \mathcal{A}^0 is a sheaf, there exists a global section $\varphi \in \mathcal{A}^0(M_0)$ such that c_{λ_0} is the restriction of φ to $U_{\lambda_0}^0$ for all $\lambda_0 \in I_0$. Now, Equation (5.1) is equivalent to $\tilde{\varepsilon}_1^* \varphi = \tilde{\varepsilon}_0^* \varphi$. We have thus proved:

Proposition 5.1. *Let \mathcal{A}^\bullet be an abelian sheaf on a simplicial space M_\bullet and let \mathcal{U}_\bullet be an open cover of M_\bullet . Then*

$$\check{H}^0(M_\bullet; \mathcal{A}^\bullet) = H^0(\mathcal{U}_\bullet; \mathcal{A}^\bullet) = \Gamma_{\text{inv}}(\mathcal{A}^\bullet) := \text{Ker}(\mathcal{A}^0(M_0) \rightrightarrows \mathcal{A}^1(M_1)).$$

(Of course, in the case of a groupoid and an abelian G -sheaf, a section in $\mathcal{A}^0(G_0)$ is in $\Gamma_{\text{inv}}(\mathcal{A}^\bullet)$ if and only if it is an invariant section in the usual sense, i.e. under the action of G .)

5.2. The group \check{H}^1 . Consider a groupoid G . The cocycle relation in degree 1 is

$$\tilde{\varepsilon}_0^* c_{\lambda_1 \lambda_2 \lambda_{12}} - \tilde{\varepsilon}_1^* c_{\lambda_0 \lambda_2 \lambda_{02}} + \tilde{\varepsilon}_2^* c_{\lambda_0 \lambda_1 \lambda_{01}} = 0$$

on $U_{\lambda_0 \lambda_1 \lambda_2 \lambda_{01} \lambda_{02} \lambda_{12} \lambda_{012}}^2$. Exactly the same method as in the preceding paragraph shows that $c_{\lambda_0 \lambda_1 \lambda_{01}}$ does not depend on the choice of λ_{01} , hence there exists a section $\varphi_{\lambda_0 \lambda_1} \in \mathcal{A}^1(\tilde{\varepsilon}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_1}^0))$ such that $c_{\lambda_0 \lambda_1 \lambda_{01}}$ is the restriction to $U_{\lambda_0 \lambda_1 \lambda_{01}}^1$ of $\varphi_{\lambda_0 \lambda_1}$. The cocycle relation becomes

$$(5.2) \quad \tilde{\varepsilon}_0^* \varphi_{\lambda_1 \lambda_2} - \tilde{\varepsilon}_1^* \varphi_{\lambda_0 \lambda_2} + \tilde{\varepsilon}_2^* \varphi_{\lambda_0 \lambda_1} = 0.$$

Coboundaries are cocycles of the form $\varphi_{\lambda_0 \lambda_1} = \tilde{\varepsilon}_0^* \alpha_{\lambda_1} - \tilde{\varepsilon}_1^* \alpha_{\lambda_0}$.

Proposition 5.2. *Let G be a topological groupoid and A be a G -module. Denote by \mathcal{A}^\bullet the associated sheaf on G_\bullet . Then $H^1(G_\bullet; \mathcal{A}^\bullet)$ is the group of G -equivariant locally trivial A -principal bundles over G_0 .*

Proof. This is well-known. Let us give the proof for completeness. Suppose we are given such a principal bundle $E \rightarrow G_0$. Choose an open cover $\mathcal{U}_0 = (U_i^0)_{i \in I_0}$ of G_0 such that for all i there exists a (not necessarily equivariant) section $x \mapsto \sigma_i(x)$ of the bundle $E \rightarrow G_0$ over the open subset U_i^0 . Let us define $\varphi_{\lambda_0 \lambda_1}(g) \in A_{r(g)}$ (for all $g \in \tilde{\varepsilon}_0^{-1}(U_{\lambda_0}^0) \cap \tilde{\varepsilon}_1^{-1}(U_{\lambda_1}^0)$) by

$$(5.3) \quad \sigma_{\lambda_0}(r(g)) = g\sigma_{\lambda_1}(s(g)) + \varphi_{\lambda_0 \lambda_1}(g).$$

(We have denoted by “+” the action of A on the A -principal bundle E .) If $(g, h) \in G_2$, then we get

$$(5.4) \quad \sigma_{\lambda_1}(r(h)) = h\sigma_{\lambda_2}(s(h)) + \varphi_{\lambda_1 \lambda_2}(h)$$

$$(5.5) \quad \sigma_{\lambda_0}(r(gh)) = gh\sigma_{\lambda_2}(s(gh)) + \varphi_{\lambda_0 \lambda_2}(gh).$$

Substitute (5.4) in (5.3) and compare to (5.5) to get the cocycle relation

$$(5.6) \quad \varphi_{\lambda_0 \lambda_2}(gh) = g\varphi_{\lambda_1 \lambda_2}(h) + \varphi_{\lambda_0 \lambda_1}(g)$$

(which is exactly (5.2)).

If σ'_i is another section, let $\alpha_i(x) = \sigma_i(x) - \sigma'_i(x)$, then from (5.3) and its analogue for σ' , we get by subtraction $\alpha_{\lambda_0}(r(g)) = g\alpha_{\lambda_1}(s(g)) + \varphi_{\lambda_0 \lambda_1}(g) - \varphi'_{\lambda_0 \lambda_1}(g)$, i.e. $\varphi' - \varphi = d\alpha$.

The above shows that any G -equivariant A -principal bundle which is trivial over each open set U_i^0 defines an element of $H^1(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$. Moreover, passing to a finer cover obviously doesn't change the Čech cohomology class, hence any bundle as above defines a cohomology class in $\check{H}^1(G_\bullet; \mathcal{A}^\bullet)$.

Conversely, suppose we are given $(\varphi_{\lambda_0 \lambda_1})$ satisfying the cocycle relation (5.6). Define

$$E = (\coprod_{i \in I_0} U_i^0 \times_{G_0} A) / \sim$$

with the identifications $(\lambda_0, x, a) \sim (\lambda_1, x, a + \varphi_{\lambda_0 \lambda_1}(x))$ and action

$$g \cdot [(\lambda_1, s(g), a)] = [(\lambda_0, r(g), a - \varphi_{\lambda_0 \lambda_1}(g))].$$

It is elementary to check that E is a locally trivial G -equivariant A -principal bundle, and that the associated 1-cocycle is indeed φ . \square

5.3. The group \check{H}^2 , extensions and the Brauer group. Let G be a topological groupoid, and let A be a G -module. Let us denote by $\text{ext}(G, A)$ the set of extensions of the form

$$A \xhookrightarrow{i} E \xrightarrow{\pi} G$$

such that the unit spaces of the groupoids A , E and G are all equal to G_0 , the maps i and π are the identity map on G_0 , and such that for all $\gamma \in E$ and all $a \in A_{s(\gamma)}$ we have

$$\gamma a \gamma^{-1} = \pi(\gamma) \cdot a.$$

In $\pi(\gamma) \cdot a$, the dot denotes the action of G on the G -module A .

Two such extensions $A \rightarrow E \rightarrow G$ and $A \rightarrow E' \rightarrow G$ are considered equivalent if there is commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & E & \longrightarrow & G \\ \downarrow \text{Id} & & \downarrow & & \downarrow \text{Id} \\ A & \longrightarrow & E' & \longrightarrow & G \end{array}$$

such that the map $E \rightarrow E'$ is a groupoid isomorphism.

There is a canonical extension in $\text{ext}(G, A)$: let $E = A \times_{p,r} G = \{(a, g) \in A \times G \mid p(a) = r(g)\}$. The source and range maps in E are $s(a, g) = s(g)$, $r(a, g) = r(g)$. The product is $(a, g)(b, h) = (a + g \cdot b, gh)$ (defined whenever $s(g) = r(g)$; the product in A is written additively). The inclusion $A \hookrightarrow E$ is $i(a) = (a, p(a))$ and the projection $\pi: E \rightarrow G$ is $\pi(a, g) = g$.

Let us call this extension the *strictly trivial* extension.

Proposition 5.3. *Let $A \xrightarrow{i} E \xrightarrow{\pi} G$ be an element of $\text{ext}(G, A)$. The following are equivalent:*

- (i) *the extension is strictly trivial;*
- (ii) *there exists a groupoid morphism $\sigma: G \rightarrow E$ which is a section of π ;*
- (iii) *there exists $\varphi: E \rightarrow A$ such that $\varphi(a\gamma) = a\varphi(\gamma)$ for all $(a, \gamma) \in A \times_{p,r} E$ and $\varphi(\gamma_1\gamma_2) = \varphi(\gamma_1) + \pi(\gamma_1) \cdot \varphi(\gamma_2)$ for all composable pairs $(\gamma_1, \gamma_2) \in E^2$.*

Proof. Let us sketch the easy proof.

(i) \implies (ii): take $\sigma(g) = (r(g), g)$.

For (ii) \implies (iii), let $\varphi(\gamma) = \gamma[\sigma \circ \pi(\gamma)]^{-1}$.

To prove (iii) \implies (i), the map $\gamma \mapsto (\varphi(\gamma), \pi(\gamma))$ is a groupoid isomorphism from E onto $A \times_{p,r} G$. \square

The set $\text{ext}(G, A)$ is an abelian group. The (“Baer”) sum $E_1 \oplus E_2$ of two extensions $A \rightarrow E_i \rightarrow G$ is given by the extension $A \rightarrow E \rightarrow G$ with

$$E = \{(\gamma_1, \gamma_2) \in E_1 \times E_2 \mid \pi_1(\gamma_1) = \pi_2(\gamma_2)\} / \sim$$

where $(a\gamma_1, \gamma_2) \sim (\gamma_1, a\gamma_2)$ if $\pi_1(\gamma_1) = \pi_2(\gamma_2)$ and $p(a) = r(\gamma_1)$. The map $\pi: E \rightarrow G$ is given by $\pi(\gamma_1, \gamma_2) = \pi_1(\gamma_1) = \pi_2(\gamma_2)$, and the inclusion $i: A \rightarrow E$ is

$$i(a) = (i_1(a), p(a)) \sim (p(a), i_2(a)).$$

Finally, the inverse of the extension is

$$A \xrightarrow{i'} \bar{E} \xrightarrow{\pi} G$$

where $\bar{E} = E$ as a groupoid, but $i'(a) = i(-a)$. Denoting by $\bar{\gamma}$ the element in \bar{E} which is the same as the element $\gamma \in E$, this means that $\overline{a\gamma} = (-a)\bar{\gamma}$. To check that $E \oplus \bar{E}$ is strictly trivial, just note that for all $g \in G$, the element $(\gamma, \bar{\gamma}) \in E \oplus \bar{E}$, does not depend on the choice of $\gamma \in E$ such that $\pi(\gamma) = g$, since $(a\gamma, \overline{a\gamma}) = (a\gamma, (-a)\bar{\gamma}) \sim (\gamma, \bar{\gamma})$. Therefore, $g \mapsto \sigma(g) = (\gamma, \bar{\gamma})$ defines a cross-section of π .

Example 5.4. When $A = G_0 \times \mathbb{T}$, and G does not act on \mathbb{T} , we obtain the group $\text{Tw}(G)$ of twists of G [10].

It is clear that $\text{ext}(G, A)$ is covariant with respect to G -module morphisms. It is also contravariant with respect to groupoid morphisms. Indeed, let $f: G' \rightarrow G$ be a groupoid morphism, and let $A' = f^*A = \{(x, a) \in G'_0 \times A \mid f(x) = p(a)\}$. Then A' is a G' -module with respect to the action $g' \cdot (x, a) = (r(g'), f(g') \cdot a)$, and there is a “pull-back” morphism

$$f^*: \text{ext}(G, A) \rightarrow \text{ext}(G', f^*A)$$

defined as follows. Let $A \rightarrow E \rightarrow G$ be an element of $\text{ext}(G, A)$, then its pull-back by f is the extension

$$A \xrightarrow{i'} E' \xrightarrow{\pi'} G'$$

where $E' = \{(\gamma, g') \in E \times G' \mid \pi(\gamma) = f(g')\}$, $\pi'(\gamma, g') = g'$, $i'(a) = (i(a), p(a))$. The groupoid structure on E' is the one induced from the product groupoid $E \times G'$.

More generally, suppose that $G'_0 \xleftarrow{\rho} Z \xrightarrow{\sigma} G_0$ is a generalized morphism from G' to G (see 2.3). Put $A' = Z \times_G A := \{(z, a) \in Z \times A \mid \sigma(z) = p(a)\} / \sim$, where $(zg, g^{-1}a) \sim (z, a)$ for all triples $(z, a, g) \in Z \times A \times G$ such that $\sigma(a) = p(a) = r(g)$. It is obvious that A' is a G' -module with sum $(z, a) + (z, b) = (z, a + b)$ and left G' -action $g'(z, a) = (g'z, a)$.

The slight defect of the group $\text{ext}(G, A)$ is that it is not invariant by Morita equivalence. To remedy this, let us define

Definition 5.5.

$$\text{Ext}(G, A) = \lim_{\mathcal{U}} \text{ext}(G[\mathcal{U}], A[\mathcal{U}])$$

where \mathcal{U} runs over open covers of G_0 (see notation (2.2)).

By construction, the group $\text{Ext}(G, A)$ is invariant under Morita equivalence (see Proposition 2.2).

Let us now come to the relation between 2-cohomology and extensions:

Proposition 5.6. *Let G be a topological groupoid, A a G -module, and \mathcal{A}^\bullet the associated sheaf over G_\bullet .*

(a) *For each open cover \mathcal{U}_\bullet of G_\bullet , there is a canonical group isomorphism*

$$(5.7) \quad \text{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0]) \cong H^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet),$$

where $\text{ext}_{\mathcal{U}}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ denotes the subgroup of elements of $\text{ext}(G[\mathcal{U}_0], A[\mathcal{U}_0])$ consisting of extensions $A[\mathcal{U}_0] \rightarrow E \xrightarrow{\pi} G[\mathcal{U}_0]$ such that π admits a continuous lifting over each open set $U_\lambda^1 \subset G_1$ ($\lambda \in \Lambda_1$).

(b) *(5.7) induces an isomorphism*

$$\text{Ext}(G, A) \cong \check{H}^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet).$$

Proof. As in the previous subsection, one easily sees that a 2-cocycle in $Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ is given by a family

$$\varphi = (\varphi_{\lambda_0 \lambda_1 \lambda_2 \lambda_{01} \lambda_{02} \lambda_{12}})$$

such that each term is a continuous function $(g, h) \mapsto \varphi_\lambda(g, h) \in A_{r(g)}$, defined on the set of pairs (g, h) such that $r(g) \in U_{\lambda_0}^0$, $s(g) \in U_{\lambda_1}^0$, $s(h) \in U_{\lambda_2}^0$, $g \in U_{\lambda_{01}}^1$, $gh \in U_{\lambda_{02}}^1$, $h \in U_{\lambda_{12}}^1$. The φ 's satisfy the cocycle identity

$$(5.8) \quad g\varphi_{\lambda_1 \lambda_2 \lambda_3 \lambda_{12} \lambda_{13} \lambda_{23}}(h, k) - \varphi_{\lambda_0 \lambda_2 \lambda_3 \lambda_{02} \lambda_{03} \lambda_{23}}(gh, k) \\ + \varphi_{\lambda_0 \lambda_1 \lambda_3 \lambda_{01} \lambda_{03} \lambda_{13}}(g, hk) - \varphi_{\lambda_0 \lambda_1 \lambda_2 \lambda_{01} \lambda_{02} \lambda_{12}}(g, h) = 0$$

Let us consider a cover \mathcal{V}_\bullet of $G[\mathcal{U}]$, $\mathcal{V}_n = (V_j^n)_{j \in J_n}$, such that

- $J_0 = \{\text{pt}\}$ and \mathcal{V}_0 is the cover consisting of the unique open set $\coprod_{i \in I_0} U_i^0$;
- $J_1 = I_0 \times I_0 \times I_1$ with $V_{ijk}^1 = \{(i, g, j) \mid r(g) \in U_i^0, s(g) \in U_j^0, g \in U_k^1\}$;
- V^n arbitrary $\forall n \geq 2$.

Consider the group $Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$, where \mathcal{A}'^\bullet is the pull-back of the sheaf \mathcal{A}^\bullet by $G[\mathcal{U}] \rightarrow G$. As above, it consists of families $\psi_{\mu_{01} \mu_{02} \mu_{12}}$ satisfying the cocycle identity

$$(5.9) \quad g\psi_{\mu_{12} \mu_{13} \mu_{23}}(h, k) - \psi_{\mu_{02} \mu_{03} \mu_{23}}(gh, k) \\ + \psi_{\mu_{01} \mu_{03} \mu_{13}}(g, hk) - \psi_{\mu_{01} \mu_{02} \mu_{12}}(g, h) = 0$$

We show that $Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$, where \mathcal{A}'^\bullet is the pull-back of the sheaf \mathcal{A}^\bullet by $G[\mathcal{U}] \rightarrow G$.

In one direction, let $\psi \in Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$. For all $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12})$ in $I_0^3 \times I_1^3$, define

$$(5.10) \quad \begin{aligned} \mu_{01} &= (\lambda_0, \lambda_1, \lambda_{01}) \\ \mu_{02} &= (\lambda_0, \lambda_2, \lambda_{02}) \\ \mu_{12} &= (\lambda_1, \lambda_2, \lambda_{12}) \end{aligned}$$

and $\varphi_\lambda = \psi_{\mu_{01} \mu_{02} \mu_{12}}$.

In the other direction, if we are given a 2-cocycle $\varphi \in Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$, we want to define a 2-cocycle $\psi \in Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$. Given $\mu = (\mu_{01}, \mu_{02}, \mu_{12}) \in J_1$, write $\mu_{ab} = (i_{ab}, j_{ab}, k_{ab})$. Then $V_\mu^1 \neq \emptyset$ implies that $j_{01} = i_{12}$, $i_{01} = i_{02}$, $j_{02} = j_{12}$, hence there exists

$$\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_{01}, \lambda_{02}, \lambda_{12}) \in I_0^3 \times I_1^3$$

such that (5.10) holds. We then define $\psi_\mu = \varphi_\lambda$.

Comparing (5.8) and (5.9), we see that $Z^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong Z^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$. Moreover, it is not hard to check that this induces an isomorphism

$$(5.11) \quad H^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong H^2(\mathcal{V}_\bullet; \mathcal{A}'^\bullet).$$

To prove the first part of the proposition, then, we can (after passing to the groupoid $G[\mathcal{U}_0]$), suppose that \mathcal{U}_0 consists of the unique open set G_0 .

Consider an extension in $\text{ext}_{\mathcal{U}}(G, A)$

$$A \hookrightarrow E \twoheadrightarrow G.$$

For each $i \in I_1$, consider a continuous section $\sigma_i: U_i^1 \rightarrow E$. Define a cochain φ by the equation

$$(5.12) \quad \sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h) = \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h)\sigma_{\lambda_{02}}(gh).$$

To see that it is indeed a cocycle, just write

$$(\sigma_{\lambda_{01}}(g)\sigma_{\lambda_{12}}(h))\sigma_{\lambda_{23}}(k) = \sigma_{\lambda_{01}}(g)(\sigma_{\lambda_{12}}(h)\sigma_{\lambda_{23}}(k))$$

and substitute relations like (5.12) to obtain

$$(5.13) \quad \begin{aligned} & g\varphi_{\lambda_{12}\lambda_{13}\lambda_{23}}(h, k) - \varphi_{\lambda_{02}\lambda_{03}\lambda_{23}}(gh, k) \\ & + \varphi_{\lambda_{01}\lambda_{03}\lambda_{13}}(g, hk) - \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h) = 0. \end{aligned}$$

Suppose that σ'_i is another continuous lifting and let $\alpha_i: U_i^1 \rightarrow A$ such that

$$(5.14) \quad \sigma'_i(g) = \alpha_i(g)\sigma_i(g).$$

Define φ' by

$$(5.15) \quad \sigma'_{\lambda_{01}}(g)\sigma'_{\lambda_{12}}(h) = \varphi'_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h)\sigma'_{\lambda_{02}}(gh).$$

Substituting (5.14) in (5.15) and comparing with (5.12), we find

$$(\varphi' - \varphi)_{\lambda_{01}\lambda_{02}\lambda_{12}}(gh) = g\alpha_{\lambda_{12}}(h) - \alpha_{\lambda_{02}}(gh) + \alpha_{\lambda_{01}}(g),$$

i.e. $\varphi' - \varphi = d\alpha$. This proves that an extension in $\text{ext}_{\mathcal{U}}(G, A)$ determines a unique cohomology class in $H^2(\mathcal{U}_\bullet; A^\bullet)$.

Conversely, given a cocycle $\varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}$, we want to construct an extension

$$A \rightarrow E \rightarrow G.$$

The idea is to set

$$(5.16) \quad E = \coprod_{i \in I_1} \{(a, g, i) \mid a \in A, g \in U_i^1, p(a) = r(g)\} / \sim,$$

with the product law

$$(5.17) \quad [a, g, \lambda_{01}][b, g, \lambda_{12}] = [a + g \cdot b + \varphi_{\lambda_{01}\lambda_{02}\lambda_{12}}(g, h), gh, \lambda_{02}].$$

To determine the correct equivalence relation in (5.16), we note that if $[a, x, i]$ represents a unit element in the groupoid E , then from the product law (5.17) we necessarily have $[a, x, i] = [a, x, i][a, x, i] = [2a + \varphi_{iii}(x, x), x, i]$, thus $[-\varphi_{iii}(x, x), x, i]$ must be the unit element. Using again (5.17), we necessarily have

$$[-\varphi_{iii}(r(g), r(g)), r(g), i][a, g, k] = [-\varphi_{iii}(r(g), r(g)) + a + \varphi_{ijk}(r(g), g), g, j]$$

thus we necessarily have

$$(5.18) \quad (a, g, k) \sim (-\varphi_{iii}(r(g), r(g)) + a + \varphi_{ijk}(r(g), g), g, j)$$

Conversely, we want to show that (5.18) defines an equivalence relation. We claim that $\psi_{kj}(g) = -\varphi_{iii}(r(g), r(g)) + \varphi_{ijk}(r(g), g)$ does not depend of the choice of i .

Let us denote $x=r(g)$. Apply (5.13) to (x, x, g) instead of (g, h, k) :

$$(5.19) \quad \varphi_{ijk}(x, g) - \varphi_{lmk}(x, g) + \varphi_{nmj}(x, g) - \varphi_{nli}(x, x) = 0.$$

Taking $g=x$ and $j=k=\ell=m=n$ we find

$$(5.20) \quad \varphi_{ijj}(x, x) = \varphi_{jji}(x, x) = \varphi_{iii}(x, x).$$

Taking $k=j$ and $n=\ell$ in (5.19) and using (5.20), we get

$$(5.21) \quad \varphi_{ijj}(x, g) = \varphi_{\ell\ell i}(x, x) = \varphi_{iii}(x, x).$$

Then, take $m=j$ and $n=\ell$ in (5.19) and use (5.20) and (5.21):

$$\begin{aligned} \varphi_{ijk}(x, g) - \varphi_{\ell jk}(x, g) + \varphi_{\ell j j}(x, g) - \varphi_{\ell\ell i}(x, x) &= 0 \\ \varphi_{ijk}(x, g) - \varphi_{\ell jk}(x, g) + \varphi_{\ell\ell\ell}(x, x) - \varphi_{iii}(x, x) &= 0. \end{aligned}$$

This proves our claim that ψ_{kj} is well-defined. Moreover, taking $n=\ell=i$ in (5.19) we get

$$(5.22) \quad \psi_{jk}(g) - \psi_{mk}(g) + \psi_{mj}(g) = 0.$$

It follows that

$$\begin{aligned} \psi_{jj} &= 0 \quad (\text{use (5.22) for } k=j) \\ \psi_{kj} &= -\psi_{jk} \quad (\text{use (5.22) for } m=k) \\ \psi_{jm} &= \psi_{jk} + \psi_{km}. \end{aligned}$$

Therefore, (5.18) defines an equivalence relation.

It is then elementary to check that (5.17) endows E with a groupoid structure such that the obvious extension

$$A \rightarrow E \xrightarrow{\pi} G$$

is an element of $\text{ext}(G, A)$, and that π admits a continuous lifting $\sigma_i: U_i^1 \rightarrow G$ defined by $\sigma_i(g) = [0, g, i]$; and that the associated cocycle is precisely φ . We leave these easy verifications to the reader.

To prove the second part of the proposition, we first pass to the inductive limit over all open covers \mathcal{U}_1 of G_1 (leaving \mathcal{U}_0 fixed) to find that

$$\text{ext}(G[\mathcal{U}_0], A[\mathcal{U}_0]) = \lim_{\mathcal{U}_1} H^2(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$$

and then take the inductive limit over \mathcal{U}_0 . □

Remark 5.7. By the same method we used to show (5.11), one can show that for each open cover \mathcal{U}_\bullet of G_\bullet , and each sheaf \mathcal{A}^\bullet over G_\bullet , the canonical morphism $f: G[\mathcal{U}_0] \rightarrow G$ induces an isomorphism

$$H^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet) \cong H^n(\mathcal{V}_\bullet; \mathcal{A}'^\bullet)$$

where \mathcal{V}_0 is the cover consisting of the unique open set $\coprod_{i \in I} U_i^0$ and the open cover $\mathcal{V}_n = (V_{i_0, \dots, i_n, j}^n)_{i_0, \dots, i_n, j \in I_0^{n+1} \times I_n}$ of $G[\mathcal{U}]_n$ is defined by

$$V_{i_0, \dots, i_n, j}^n = \{(i_0, \dots, i_n, g_1, \dots, g_n) \mid r(g_1) \in U_{i_0}^0, s(g_1) \in U_{i_1}^0, \dots, s(g_n) \in U_{i_n}^0, (g_1, \dots, g_n) \in U_j^n\}$$

(see Remark 2.1).

Remark 5.8. For completeness, it would remain to examine the relation between the *sheaf* 2-cohomology groups and extensions of non-paracompact groupoids, since it is not obvious whether H^2 is equal or not to \check{H}^2 in this case. However, we won't develop this, due to lack of applications.

Corollary 5.9. *If G is a locally compact Hausdorff groupoid with Haar system then $\check{H}^2(G_\bullet; \mathbb{T}) \cong \text{Ext}(G, G_0 \times \mathbb{T})$ is the Brauer group of G*

Proof. Use for instance Proposition 2.13 and remarks preceding Proposition 2.29 of [20]. \square

6. COMPARISON WITH MOORE'S COHOMOLOGY

Recall [16] that if G is a locally compact group and A is a Polish abelian group (i.e., as a topological space, A admits a separable complete metric), then A is a G -module if G acts (continuously) by automorphisms on A .

Given a Polish G -module A , let $I(A)$ be the set of μ -measurable functions from G to A (μ being the Haar measure), modulo equality almost everywhere. Then $I(A)$ is again a Polish G -module, with action $(\gamma \cdot f)(x) = \gamma f(\gamma^{-1}x)$ (caution: our definition is different but isomorphic to Moore's definition of $I(A)$).

The G -module A embeds in $I(A)$ via the obvious map

$$i_A: A \rightarrow I(A), \quad (i_A(a))(x) = a.$$

Let $U(A) = I(A)/A$. Then, using measurable cocycles, Moore defined cohomology groups $H^n(G, A)$ which are characterized by the proposition below, where $I(A)$ is defined as above and $F(A) = A^G$ (the sub-module of G -fixed points).

Proposition 6.1. *Let \mathcal{C}_1 and \mathcal{C}_2 be two abelian categories. Suppose that F is a left-exact functor from \mathcal{C}_1 to \mathcal{C}_2 , that $I: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ is a functor and $i_A: A \hookrightarrow I(A)$ is a natural injection. Then*

- (a) *there exists, up to isomorphism, at most one sequence of functors $H^n: \mathcal{C}_1 \rightarrow \mathcal{C}_2$ such that*
 - 1) $H^0 = F$
 - 2) *Any exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ induces a natural long exact sequence $0 \rightarrow H^0(A') \rightarrow H^0(A) \rightarrow H^0(A'') \xrightarrow{\partial} H^1(A') \rightarrow \dots$*
 - 3) $H^n(I(A)) = 0$ for all A and for all $n \geq 1$.

- (b) If I is an exact functor and $I(i_A)=i_{I(A)}$ for all A , then 3) may be replaced by 3') $H^1(I(A))=0 \forall A$.
(c) If moreover $F(I(A)) \rightarrow F(U(I(A)))$ is surjective for all A , where $U(A)=I(A)/A$, then there exists a sequence of functors satisfying 1), 2) and 3).

Proof. a) This is essentially [2, Theorem 1] or [16, Theorem 2]: using the long exact sequence associated to $0 \rightarrow A \xrightarrow{i_A} I(A) \rightarrow U(A) \rightarrow 0$, one gets $H^n(A) \cong H^{n-1}(U(A))$ for $n \geq 2$ and $H^1(A)=\text{coker}(F(I(A)) \rightarrow F(U(A)))$, thus H^n is uniquely determined by induction on n .

b) If I is exact then

$$0 \rightarrow I(A) \xrightarrow{I(i_A)} I(I(A)) \rightarrow I(U(A)) \rightarrow 0$$

is exact, and

$$0 \rightarrow I(A) \xrightarrow{i_{I(A)}} I(I(A)) \rightarrow U(I(A)) \rightarrow 0$$

is exact by definition. Therefore, the assumption $I(i_A)=i_{I(A)}$ implies $I(U(A))=U(I(A))$ canonically. Thus, 3') implies that for $n \geq 2$,

$$H^n(I(A))=H^1(U^{n-1}I(A))=H^1(IU^{n-1}(A))=0.$$

c) Define a resolution $0 \rightarrow A \rightarrow A_0 \rightarrow A_1 \rightarrow \dots$ by $A_0=I(A)$, and $A_{n+1}=I(A_n/A_{n-1})$. The map $A_n \rightarrow A_{n+1}$ is the composition $A_n \rightarrow A_n/A_{n-1} \xrightarrow{i} I(A_n/A_{n-1})$. Define $H^n(A)$ to be the cohomology of the complex $F(A_0) \rightarrow F(A_1) \rightarrow \dots$ and let us check that properties 1), 2) and 3') hold.

1) $H^0(A)=\text{Ker}(F(A) \rightarrow F(I(I(A)/A)))=\text{Ker}(F(I(A)) \rightarrow F(I(A)/A))$ since F preserves injectivity of morphisms, and $I(A)/A \rightarrow I(I(A)/A)$ is injective. Using left exactness of F , we see that $H^0(A)=F(\text{Ker}(I(A) \rightarrow I(A)/A))=F(A)$.

2) Since I is an exact functor, we have an exact sequence of complexes $0 \rightarrow A'_* \rightarrow A_* \rightarrow A''_* \rightarrow 0$, hence the conclusion by the Snake lemma.

3') $H^1(I(A))=F(I(I(A)))/F(U(I(A)))=0$. \square

For instance, in part a) of the proposition, if $I(A)$ is an injective object for all A then H^n are the right derived functors of F .

We are now ready to prove:

Proposition 6.2. *Let G be a locally compact group. Let A be a Polish G -module and let \mathcal{A}^\bullet be the associated sheaf on G_\bullet (see Definition 3.9 and below). Then $\check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G, A)$.*

Proof. We just need to check that $\check{H}^*(G_\bullet; \cdot)$ satisfies the conditions 1)–3) of Proposition 6.1.

1) $\check{H}^0(G_\bullet; \mathcal{A}^\bullet)=A^G$ was proved in Proposition 5.1 and 2) is Corollary 4.13.

I is an exact functor [16, Proposition 9], and it is obvious that $I(i_A)=i_{I(A)}$. It thus remains to show that $\check{H}^1(G_\bullet; \mathcal{B}^\bullet)=0$ if \mathcal{B}^\bullet is the sheaf on G_\bullet associated

to the G -module $I(A)$. Recall that a 1-cocycle φ is a continuous function $\varphi: G \rightarrow I(A)$ satisfying

$$g_1\varphi(g_2) - \varphi(g_1g_2) + \varphi(g_1) = 0$$

(see (5.6)), hence φ is a cocycle in the Moore complex. But $H_{\text{Moore}}^1(G, I(A)) = 0$, hence there exists $\psi \in I(A)$ such that $\varphi(g) = g\psi - \psi$ for all $g \in G$. Therefore, $\varphi = 0$ in $H^1(\mathcal{U}_\bullet; \mathcal{B}^\bullet)$. \square

Remark 6.3. One can easily construct explicitly the isomorphism $\check{H}^n(G_\bullet; \mathcal{A}^\bullet) \rightarrow H_{\text{Moore}}^n(G, A)$. Take $\varphi \in Z^n(\mathcal{U}_\bullet; \mathcal{A}^\bullet)$ where \mathcal{U}_\bullet is an open cover. Choose measurable maps $\theta_k: G^k \rightarrow I_k$ such that $x \in U_{\theta_k(x)}^k$ for all $x \in G^k$, and define

$$c(g_1, \dots, g_n) = \varphi_{\lambda(g_1, \dots, g_n)}(g_1, \dots, g_n)$$

where $\lambda(g_1, \dots, g_n) \in \Lambda_n$ is defined by $\lambda(g_1, \dots, g_n)(f) = \theta_r(\tilde{f}(g_1, \dots, g_n)) \in I_k$ for every $f \in \text{Hom}_{\Delta'}(k, n)$.

Remark 6.4. One might wonder if it is possible to define, for every locally compact groupoid and every (say, locally compact) G -module A , an analogue of the Moore complex, using measurable cochains $c(g_1, \dots, g_n) \in A_{r(g_1)}$. In order to get the usual cohomology groups when G is a space, one should probably use the sheaf over G_0 of functions $c(g_1, \dots, g_n) \in A_{r(g_1)}$ which are measurable in the “leaf” direction and continuous in the “transverse” direction. Since this approach doesn’t seem simpler or more useful than Čech or sheaf cohomology of simplicial spaces, we won’t develop this further.

7. COMPARISON WITH SHEAF COHOMOLOGY AND HAEFLIGER’S COHOMOLOGY

Let M_\bullet be a simplicial space and \mathcal{A}^\bullet an abelian sheaf on M_\bullet . By definition [4], the cohomology groups $H^n(M_\bullet; \mathcal{A}^\bullet)$ are the derived functors of the functor $\Gamma_{\text{inv}}(M_\bullet; \mathcal{A}^\bullet) = \text{Ker}(\mathcal{A}^0(M_0) \rightrightarrows \mathcal{A}^1(M_1))$, thus they coincide with Haefliger’s cohomology groups in the case of étale groupoids [6].

A practical way of calculating the cohomology groups is to take a resolution $(\mathcal{L}^{\bullet, q})_{q \in \mathbb{N}}$ of \mathcal{A}^\bullet such that $H^n(M_p; \mathcal{L}^{p, q}) = 0 \ \forall n \geq 1, \forall p, q \geq 0$ and take the cohomology of the double complex $(\mathcal{L}^{p, q}(M_p))$, where the first differential is $d' = \sum_{k=0}^{p+1} (-1)^k \varepsilon_k^*$ and the second differential d'' is the differential in the resolution

$$\mathcal{A}^p \rightarrow \mathcal{L}^{p, 0} \rightarrow \mathcal{L}^{p, 1} \rightarrow \dots$$

(See [4, §5.2.3] in the general case, [3, §2.7] or [6] in the case of étale groupoids).

In this section, we show:

Proposition 7.1. *Let M_\bullet be a paracompact simplicial space, and \mathcal{A}^\bullet an abelian sheaf on M_\bullet . Then $H^*(M_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(M_\bullet; \mathcal{A}^\bullet)$. In particular, $\check{H}^*(G_\bullet; \mathcal{A}^\bullet)$ are Haefliger’s cohomology groups if G is an étale paracompact groupoid and \mathcal{A}^\bullet is an abelian G -sheaf.*

We will again use Proposition 6.1. Consider $I(\mathcal{A})^\bullet$ the sheaf such that $I(\mathcal{A})^n(U)$ consists of maps f (continuous or not) from $\tilde{\varepsilon}_0^{-1}(U)$ to \mathcal{A}^{n+1} such that $f(x) \in \mathcal{A}_x^{n+1}$ for all $x \in \tilde{\varepsilon}_0^{-1}(U)$. We will need the

Lemma 7.2. *For any open cover \mathcal{U}_\bullet and all $n \geq 1$, $H^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet) = \{0\}$.*

Proof. Let us first explain the simplicial structure on the sheaf $I(\mathcal{A})^\bullet$. Given $f \in \text{Hom}_\Delta(k, n)$, $U \subset M_k$, $V \subset M_n$ such that $\tilde{f}(V) \subset U$ and a section φ of $I(\mathcal{A})^k$ over U , we have to produce a section $\tilde{f}\varphi \in \Gamma(V, I(\mathcal{A})^n)$.

Define $f' \in \text{Hom}_\Delta(k+1, n+1)$ such that

$$(7.1) \quad f'(0)=0 \text{ and } \varepsilon_0 \circ f = f' \circ \varepsilon_0: [k] \rightarrow [n+1].$$

Since $\tilde{f}(V) \subset U$ we have $\tilde{f}'(\tilde{\varepsilon}_0^{-1}(V)) \subset \tilde{\varepsilon}_0^{-1}(U)$, thus we get a section

$$x \in \tilde{\varepsilon}_0^{-1}(V) \mapsto \varphi(\tilde{f}'(x)) \in \mathcal{A}_{\tilde{f}'(x)}^{k+1} \mapsto (\tilde{f}')^* \varphi(\tilde{f}'(x)) \in \mathcal{A}_x^{n+1}.$$

Now, let us show that the complex $C^*(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$ is homotopically trivial. First, for all $n \in \mathbb{N}$ and all $x \in M_n$, let us choose $\theta(x) \in I_n$ such that $x \in U_{\theta(x)}^n$. We define a homotopy

$$H: C^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet) \rightarrow C^{n-1}(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$$

by $(H\varphi)_\lambda(x) = \tilde{\eta}_0^* \varphi_{\lambda'_x}(\tilde{\eta}_0(x))$, $\forall x \in \tilde{\varepsilon}_0^{-1}(U_\lambda) \subset M_n$, $\forall \lambda \in \Lambda_{n-1}$, and $\lambda'_x \in \Lambda_n$ is defined as follows: for all $f \in \text{Hom}_{\Delta'}(k, n)$, let

$$\lambda'_x(f) = \begin{cases} \lambda(\eta_0 \circ f) & \text{if } f(0) \neq 0 \\ \theta(\tilde{f}(x)) & \text{if } f(0) = 0 \end{cases} \in I_k.$$

Let $\varphi \in C^n(\mathcal{U}_\bullet; I(\mathcal{A})^\bullet)$. Let us compute $(dH + Hd)\varphi$ and compare it with φ . We have

$$(d\varphi)_\lambda(x) = \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_k'^* \varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_k'(x)) \in \mathcal{A}_x^{n+2}.$$

The meaning of this formula is the following: we take $x \in M_{n+2}$, then its image $\tilde{\varepsilon}_k'(x)$ (see notation (7.1)) is in M_{n+1} . Its image $\varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_k'(x))$ belongs to $\mathcal{A}_{\tilde{\varepsilon}_k'(x)}^{n+1}$ is restricted (see (3.1)) by $\tilde{\varepsilon}_k': M_{n+2} \rightarrow M_{n+1}$ to an element of \mathcal{A}_x^{n+2} .

Actually, we have $\varepsilon'_k = \varepsilon_{k+1}$ since $\varepsilon_0 \circ \varepsilon_k = \varepsilon_{k+1} \circ \varepsilon_0 : [n] \rightarrow [n+2]$, hence

$$\begin{aligned}
 (d\varphi)_\lambda(x) &= \sum_{k=0}^{n+1} (-1)^k \tilde{\varepsilon}_{k+1}^* \varphi_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_{k+1}(x)) \\
 (dH\varphi)_\lambda(x) &= \sum_{k=0}^n (-1)^k \tilde{\varepsilon}_{k+1}^* (H\varphi)_{\tilde{\varepsilon}_k(\lambda)}(\tilde{\varepsilon}_{k+1}(x)) \\
 &= \sum_{k=0}^n (-1)^k \tilde{\varepsilon}_{k+1}^* \tilde{\eta}_0^* \varphi_{\tilde{\varepsilon}_k(\lambda)'}(\tilde{\varepsilon}_{k+1}(x)) (\tilde{\eta}_0 \circ \tilde{\varepsilon}_{k+1}(x)) \\
 (Hd\varphi)_\lambda(x) &= \tilde{\eta}_0^* (d\varphi)_{\lambda'_x}(\tilde{\eta}_0(x)) \\
 &= \sum_{k=0}^{n+1} (-1)^k \tilde{\eta}_0^* \tilde{\varepsilon}_{k+1}^* \varphi_{\tilde{\varepsilon}_k(\lambda'_x)}(\tilde{\varepsilon}_{k+1} \tilde{\eta}_0(x)).
 \end{aligned}$$

In the last sum, for $k=0$ we get $\varphi_{\tilde{\varepsilon}_0(\lambda'_x)}(x)$. Now, $(\tilde{\varepsilon}_0(\lambda'_x))(f) = \lambda'_x(\varepsilon_0 \circ f) = \lambda(\eta_0 \circ \varepsilon_0 \circ f) = \lambda(f)$, thus the term for $k=0$ is just $\varphi_\lambda(x)$.

To show that the other terms in the sum $dH + Hd$ cancel out, we just need to check that for all $k \geq 1$,

- (a) $\eta_0 \circ \varepsilon_{k+1} = \varepsilon_k \circ \eta_0$, and
- (b) $\tilde{\varepsilon}_k(\lambda'_x) = \tilde{\varepsilon}_{k-1}(\lambda)'_{\tilde{\varepsilon}_k(x)}$.

Assertion (a) is straightforward. Let us prove (b).

If $f(0) \neq 0$ then $\tilde{\varepsilon}_k(\lambda'_x)(f) = \lambda'_x(\varepsilon_k \circ f) = \lambda(\eta_0 \circ \varepsilon_k \circ f) = \lambda(\varepsilon_{k-1} \circ \eta_0 \circ f) = (\tilde{\varepsilon}_{k-1}(\lambda))'_{\tilde{\varepsilon}_k(x)}(f)$.

If $f(0)=0$ then $\tilde{\varepsilon}_k(\lambda'_x)(f) = \lambda'_x(\varepsilon_k \circ f) = \theta(\widetilde{\varepsilon_k \circ f(x)}) = \theta(\tilde{f} \circ \tilde{\varepsilon}_k(x)) = ((\tilde{\varepsilon}_{k-1} \lambda)')_{\tilde{\varepsilon}_k(x)}(f)$. \square

Remark 7.3. In the case of an étale groupoid G and cohomology groups with coefficients in G -sheaves, Haefliger [6], following Atiyah and Wall in the case of discrete groups [2], characterized the cohomology groups as the unique sequence of functors H^n such that

- (a) $H^0 = \Gamma_{\text{inv}}$,
- (b) H^* admits long exact sequences, and
- (c) $H^n(G; I(\mathcal{A})) = 0$ for all $n \geq 1$ and for each G -sheaf $I(\mathcal{A})$.

Let us now prove Proposition 7.1. First, we note that $\mathcal{A}^\bullet \hookrightarrow I(\mathcal{A})^\bullet$ canonically: if $c \in \mathcal{A}^n(U)$, then $\varphi(x) = \tilde{\varepsilon}_0^*[c(\tilde{\varepsilon}_0(x))] \in \mathcal{A}_x^{n+1}$ is a section of $I(\mathcal{A})^n$ over U .

Using the uniqueness part in Proposition 6.1, it suffices to show that

$$\check{H}^n(M_\bullet; I(\mathcal{A}^\bullet)) = H^n(M_\bullet; I(\mathcal{A})^\bullet) = \{0\} \quad \forall n \geq 1.$$

This is true for \check{H}^n thanks to Lemma 7.2.

Define inductively a resolution

$$(7.2) \quad 0 \rightarrow I(\mathcal{A})^\bullet \rightarrow \mathcal{L}^{\bullet,0} \rightarrow \mathcal{L}^{\bullet,1} \rightarrow \dots$$

by $\mathcal{L}^{\bullet,0}=I(I(\mathcal{A}))^\bullet$ and $\mathcal{L}^{\bullet,q+1}=I(\mathcal{L}^{\bullet,q}/\mathcal{L}^{\bullet,q-1})$. Since $\mathcal{L}^{p,q}$ is flabby for all $p, q \geq 0$, the double complex $K=(\mathcal{L}^{p,q}(M_p))$ computes $H^*(M_\bullet, I(\mathcal{A})^\bullet)$ (see above the introduction of this section).

The E_2 -term with respect to the first filtration is $E_2^{p,q}=H^p H^q(K)$. Since

$$0 \rightarrow I(\mathcal{A})^p \rightarrow \mathcal{L}^{p,0} \rightarrow \mathcal{L}^{p,1} \rightarrow \dots$$

is an exact sequence of flabby sheaves,

$$0 \rightarrow \Gamma(M_p; I(\mathcal{A})^p) \rightarrow \Gamma(M_p; \mathcal{L}^{p,0}) \rightarrow \dots$$

is exact, hence $E_2^{p,q}=0$ for $q \geq 1$ and $E_2^{p,0}=H^p(\Gamma(M_*; I(\mathcal{A})^\bullet))$. Using again Lemma 7.2 for the cover $\mathcal{U}_n=\{M_n\}$, we get $E_2^{p,0}=0$ for $p \geq 1$. Finally, $H^n(M_\bullet; I(\mathcal{A})^\bullet)=0$ for all $n \geq 1$.

8. INVARIANCE BY MORITA EQUIVALENCE

Let G be a topological groupoid and \mathcal{A}^\bullet be an abelian sheaf on G_\bullet . We will show that $H^*(G_\bullet; \mathcal{A}^\bullet)$ and $\check{H}^*(G_\bullet; \mathcal{A}^\bullet)$ are invariant under Morita equivalence.

More precisely, if G' is another groupoid and \mathcal{A}'^\bullet is a sheaf on G'_\bullet , we say that (G, \mathcal{A}^\bullet) is Morita equivalent to $(G', \mathcal{A}'^\bullet)$ if there exists a groupoid G'' , a sheaf \mathcal{A}''^\bullet on G''_\bullet and (continuous) groupoid morphisms

$$G \xleftarrow{f} G'' \xrightarrow{f'} G'$$

such that f and f' are Morita equivalences and $\mathcal{A}''^\bullet \cong f^* \mathcal{A}^\bullet \cong f'^* \mathcal{A}'^\bullet$. Then

Proposition 8.1. *With the above assumptions, f and f' induce isomorphisms in sheaf and Čech cohomology, thus*

$$H^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G'_\bullet; \mathcal{A}'^\bullet) \quad \text{and} \quad \check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(G'_\bullet; \mathcal{A}'^\bullet).$$

Proof. By standard arguments (compare with Proposition 2.2), it suffices to show that for any open cover $\mathcal{U}=(U_i)_{i \in I}$ of G_0 , the canonical morphism $f: G[\mathcal{U}] \rightarrow G$ induces isomorphisms $H^*(G_\bullet; \mathcal{A}^\bullet) \cong H^*(G[\mathcal{U}]_\bullet; f^* \mathcal{A}^\bullet)$ and $\check{H}^*(G_\bullet; \mathcal{A}^\bullet) \cong \check{H}^*(G[\mathcal{U}]_\bullet; f^* \mathcal{A}^\bullet)$. Below, we will abusively write $H^*(G[\mathcal{U}]_\bullet; \mathcal{A}^\bullet)$ instead of $H^*(G[\mathcal{U}]_\bullet; f^* \mathcal{A}^\bullet)$.

For Čech cohomology, using Remark 5.7, we have

$$(8.1) \quad \check{H}^n(G_\bullet; \mathcal{A}^\bullet) = \lim_{\mathcal{V}} \lim_{\mathcal{W}_\bullet} H^n(\mathcal{W}_\bullet; \mathcal{A}^\bullet)$$

where $\mathcal{V}=(V_j)_{j \in J}$ runs over open covers of G_0 and \mathcal{W}_\bullet runs over open covers of $G[\mathcal{V}]$ such that \mathcal{W}_0 consists of the single open set $\coprod V_j$.

Similarly,

$$(8.2) \quad \check{H}^n(G[\mathcal{U}]_\bullet; \mathcal{A}^\bullet) = \lim_{\mathcal{V}'} \lim_{\mathcal{W}'_\bullet} H^n(\mathcal{W}'_\bullet; \mathcal{A}^\bullet).$$

where $\mathcal{V}'=(V'_j)_{j \in J'}$ runs over open covers of $G[\mathcal{U}]_0$ and \mathcal{W}'_\bullet runs over open covers of $G[\mathcal{U}][\mathcal{V}']$ such that \mathcal{W}'_0 consists of the single open set $\coprod V'_j$.

Now, note that if \mathcal{V}' is an open cover of $G[\mathcal{U}]_0$ which is finer than the cover $(\{i\} \times U_i)_{i \in I}$, then there exists an open cover \mathcal{V} of G_0 such that

$G[\mathcal{V}] \cong G[\mathcal{U}][\mathcal{V}']$ (the elementary proof is left to the reader). Therefore, in the right hand sides of (8.1) and (8.2), the terms $\lim_{\mathcal{W}_\bullet} H^n(\mathcal{W}_\bullet; \mathcal{A}^\bullet)$ and $\lim_{\mathcal{W}'_\bullet} H^n(\mathcal{W}'_\bullet; \mathcal{A}^\bullet)$ are identical. It follows that the right hand sides of (8.1) and (8.2) are equal, hence Čech cohomology is invariant by Morita equivalence.

From the above, we already find that sheaf cohomology is invariant under Morita-equivalence when the groupoid is paracompact. In fact, this holds for a general topological groupoid. Let us sketch the proof for completeness.

Consider the resolution

$$\mathcal{A}^\bullet \rightarrow \mathcal{L}^{\bullet,0} \rightarrow \mathcal{L}^{\bullet,1} \rightarrow \dots$$

constructed like (7.2). Since $\mathcal{L}^{p,q}$ is flabby for all p, q , the double complex $(\mathcal{L}^{p,q}(G_p))$ computes $H^*(G_\bullet; \mathcal{A}^\bullet)$ and since the lines are exact (7.2), $H^*(G_\bullet; \mathcal{A}^\bullet)$ is the cohomology of the complex $(\Gamma_{\text{inv}}(\mathcal{L}^{\bullet,q}))_{q \in \mathbb{N}}$.

Similarly, $H^*(G[\mathcal{U}]_\bullet; \mathcal{A}^\bullet)$ is the cohomology of the complex $(\Gamma_{\text{inv}}(f^* \mathcal{L}^{\bullet,q}))_{q \in \mathbb{N}}$. Now, it is elementary to check that for every sheaf \mathcal{B}^\bullet , $\Gamma_{\text{inv}}(\mathcal{B}^\bullet)$ is isomorphic to $\Gamma_{\text{inv}}(f^* \mathcal{B}^\bullet)$. \square

REFERENCES

- [1] Artin, M. et. al., Théorie des topos et cohomologie étale des schémas, Séminaire de géométrie algébrique, *Lecture Notes in Mathematics* **269**, **270**, **305** (Springer, 1972-1973).
- [2] Atiyah, M. F.; Wall, C. T. C. Cohomology of groups. 1967. *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965) pp. 94–115 Thompson, Washington, D.C.
- [3] Crainic, M. and Moerdijk, I. A homology theory for étale groupoids. *J. Reine Angew. Math.* **521** (2000), 25–46.
- [4] Deligne, P. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.* No. **44** (1974), 5–77.
- [5] Godement, R. *Topologie algébrique et théorie des faisceaux*. Hermann, Paris. Third Edition (1973).
- [6] Haefliger, A. Differential cohomology. *Differential topology (Varenna, 1976)*, pp. 19–70, Liguori, Naples, 1979.
- [7] Haefliger, A., Groupoïdes d’holonomie et classifiants, Structure transverse des feuilletages, Toulouse 1982, *Astérisque* **116** (1984), 70–97.
- [8] Hilsen, M., and Skandalis, G., Morphismes K -orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes). *Ann. Sci. École Norm. Sup.* **20** (1987), 325–390.
- [9] Kumjian, A. On equivariant sheaf cohomology and elementary C^* -bundles. *J. Operator Theory* **20** (1988), no. 2, 207–240.
- [10] Kumjian, A., Muhly, P., Renault, J. and Williams, D., The Brauer group of a locally compact groupoid, *Amer. J. Math.* **120** (1998), 901–954.
- [11] Landsman, N. P. Quantized reduction as a tensor product. Quantization of singular symplectic quotients, 137–180, *Progr. Math.*, **198**, Birkhäuser, Basel, 2001.
- [12] Le Gall, P.-Y., Théorie de Kasparov équivariante et groupoïdes, *K-Theory* **16** (1999), 361–390.
- [13] Moerdijk, I. Classifying toposes and foliations. *Ann. Inst. Fourier (Grenoble)* **41** (1991), no. 1, 189–209.

- [14] Moerdijk, I. Lie groupoids, gerbes, and non-abelian cohomology. *K-Theory* **28** (2003), no. 3, 207–258.
- [15] Moore, C. Extensions and low dimensional cohomology theory of locally compact groups. I, II. *Trans. Amer. Math. Soc.* **113** (1964), 40–63; *ibid.*, 64–86.
- [16] Moore, C. Group extensions and cohomology for locally compact groups. III, IV. *Trans. Amer. Math. Soc.* **221**, No. 1 (1976), 1–33; *ibid.*, 35–58.
- [17] Mrčun, J., Functoriality of the bimodule associated to a Hilsum-Skandalis map, *K-Theory* **18** (1999), 235–253.
- [18] Renault, J. A groupoid approach to C^* -algebras. *Lecture Notes in Mathematics*, **793**. Springer, Berlin, 1980.
- [19] Segal, G. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.* No. **34**, (1968), pp 105–112.
- [20] Tu, J.L., Xu, P. and Laurent-Gengoux, C. Twisted K -theory of differentiable stacks. [math.KT/0306138](#).